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lim sup and lim inf

Let $\{s_n\} \subseteq \mathbb{R}$ be a sequence bounded above. Then $\forall N$

$$v_N := \sup\{s_n \mid n \geq N\} \text{ exists.}$$

Since $\{s_n \mid n \geq N\} \supseteq \{s_n \mid n \geq N+1\}$

$$v_N = \sup\{s_n \mid n \geq N\} \geq \sup\{s_n \mid n \geq N+1\} = v_{N+1}$$

Hence $\{v_N\}$ is decreasing. $\Rightarrow$ $\{v_N\}$ either converges or diverges to $-\infty$.

We define

$$\boxed{\limsup s_n := \lim v_N \qquad \text{(which may be } -\infty)}$$

Ex $s_n = -n$, $v_N = \sup\{-n \mid n \geq N\} = -N$ so $\limsup(-n) = -\infty$.

$s_n = (-1)^n$ $\sup\{(-1)^n \mid n \geq N\} = 1$ $\limsup(-1)^n = 1$

Similarly if $\{s_n\} \subseteq \mathbb{R}$ is bounded below, we define

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty}\left(\inf\{s_n \mid n \geq N\}\right).$$

Ex $s_n = (-1)^n$ $\inf\{(-1)^n \mid n \geq N\} = -1$ $\liminf_{n\to\infty}(-1)^n = -1$

Remark $\inf\{s_n \mid n \geq N\} \leq s_N \leq \sup\{s_n \mid n \geq N\}$. $\forall N$.

Remark Given an arbitrary sequence $\{s_n\}$, the sets $\{s_n \mid n \geq N\}$ need not be bounded above.

Then $\sup\{s_n \mid n \geq N\} = +\infty$ and we define

$$\limsup s_n = +\infty.$$

Similarly if $\inf\{s_n \mid n \geq N\} = -\infty$ we define

$$\liminf s_n = -\infty.$$

Ex $s_n = (-1)^n n$. $\limsup s_n = +\infty$, $\liminf s_n = -\infty$.

<u>Theorem 7.1</u> Let $\{s_n\}$ be a sequence in $\mathbb{R}$

(a) if $\{s_n\}$ converges or if it diverges to $\pm\infty$ Then

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(b) If $\liminf s_n = \limsup s_n$ (both could be $+\infty$ or $-\infty$)

Then $\lim s_n = \liminf s_n = \limsup s_n$.

<u>Proof</u> (we only consider the case where limits are real)

(a) Let $L = \lim s_n$. Then $\forall \varepsilon > 0$ $\exists N$ st for $n > N$ $|s_n - L| < \varepsilon/2$ equivalently

$$L - \varepsilon/2 < s_n < L + \varepsilon/2$$

Then $\forall M > N$

(*) $L - \varepsilon < L - \varepsilon/2 \le \inf\{s_n \mid n \ge M\} \le \sup\{s_n \mid n \ge N\} \le L + \varepsilon/2 < L + \varepsilon$

$$\Rightarrow \lim(\inf\{s_n \mid n \ge M\}) = L = \lim(\sup\{s_n \mid n \ge M\})$$

(b) If $\liminf s_n = L = \limsup s_n$,

Then $\forall \varepsilon > 0$ $\exists N$ st for $M > N$

$$L - \varepsilon < \inf\{s_n \mid n \ge M\} \le s_M \le \sup\{s_n \mid n \ge M\} < L + \varepsilon$$

$\Rightarrow s_n \to L$. $\square$

<u>Completeness.</u>

<u>Definition</u> A sequence $\{s_n\}$ in a metric space (E, d) is <u>Cauchy</u> if $\forall \varepsilon > 0$ $\exists N$ so that $n, m > N \Longrightarrow d(s_n, s_m) < \varepsilon$.

<u>Lemma 7.2</u> Any convergent sequence $\{s_n\}$ in (E, d) is Cauchy.

<u>Proof</u> Suppose $s_n \to L$. Then given $\varepsilon > 0$ $\exists N$ st $n > N \Rightarrow d(s_n, L) < \varepsilon/2$. And then for $n, m > N$

$$d(s_n, s_m) \le d(s_n, L) + d(L, s_m) < \varepsilon/2 + \varepsilon/2.$$

<u>Example</u> Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $\{s_n\}$ is <u>not</u> Cauchy

Consequently $\{S_n\}$ does not converge (and since $\{S_n\}_1^\infty$ is increasing it has to diverge to $+\infty$, ie $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$)

Proof

$$S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

Therefore, if $\{S_n\}$ were Cauchy, there would N st for all $m \geq n > N$

$|S_m - S_n| < 1/2$, which is impossible. □

There are metric spaces where a Cauchy sequence may not have a limit.

Ex $E = \mathbb{R} \setminus \{0\}$, $d(x,y) = |x-y|$ $S_n = \frac{1}{n}$ is Cauchy (since $\{S_n\}$ converges in $\mathbb{R}$) but $\{S_n\}$ has no limit in $\mathbb{R} \setminus \{0\}$

Definition A metric space is complete if every Cauchy sequence converges.

Ex $\mathbb{Q}$ is not complete: $\forall x \in \mathbb{R}$ $(x - 1/n, x + 1/n)$ contains a rational number s_n. Hence $s_n \to x$ in $\mathbb{R}$ $\Rightarrow$ $\{s_n\}$ is Cauchy but

$x = \lim s_n \notin \mathbb{Q}$

We'll prove: (i) $\mathbb{R}$ is complete, (ii) $\mathbb{R}^n$ is complete $\forall n$

(iii) Any closed subset of a complete space is complete.

Lemma 7.3 Let (E,d) be a metric space, $\{s_n\} \subseteq E$ Cauchy.

Then $\{s_n\}$ is bounded.

Proof Since $\{s_n\}$ is Cauchy $\exists N$ st $\forall m,n \geq N \Rightarrow d(s_n, s_m) < 1$

Let $r = \max\{d(s_1, s_N), \ldots d(s_{N-1}, s_N)\} + 1$

Then $\forall k$ $d(s_N, s_k) < r$ i.e. $s_k \in B_r(s_N)$. □

Lemma 7.4 Suppose $\{s_n\}$ is Cauchy (in some metric space (E,d))

$\{s_{n_k}\}$ a subsequence with $s_{n_k} \xrightarrow[k \to \infty]{} L$.

Then $s_n \to L$.

Proof Since $s_{n_k} \to L$ $\forall \varepsilon > 0$ $\exists K$ st $d(s_{n_k}, L) < \varepsilon/2$ for $k \geq K$.
Since $\{s_n\}$ is Cauchy $\exists N$ st $d(s_n, s_m) < \varepsilon/2$ for $n, m \geq N$.
Then for $n > M := \max(N, K)$

$d(s_n, s_{n_M}) < \varepsilon/2$ since $n_M \geq M \geq N$, and $d(s_{n_M}, L) < \varepsilon/2$ since $n_M \geq M \geq K$

And then

$$d(L, s_n) \leq d(s_n, s_{n_M}) + d(s_{n_M}, L) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$

Lemma 7.5 Any closed subset C of a complete metric space (E, d) is complete.

Proof Let $\{s_n\} \subseteq C$ be a Cauchy sequence (w.r.t. d). Then $\{s_n\}$ is Cauchy in $\underline{E}$. Since E is complete, $\{s_n\}$ converges to some $L \in E$. Since $\{s_n\} \subseteq C$ and C is closed, $L = \lim s_n \in C$.
$\therefore$ C is complete.

Remains to prove: $\mathbb{R}$ and $\mathbb{R}^n$ are complete. Let's start with $\mathbb{R}$.
Suppose $\{s_n\} \subset \mathbb{R}$ is Cauchy. Then by 7.3 $\{s_n\}$ is bounded.
Consider

$$X = \{ x \in \mathbb{R} \mid \exists \text{ subsequence } \{s_{n_k}\} \text{ in } \{s_n\} \text{ with } s_{n_k} \to x \}$$

If we know that $X \neq \emptyset$, we're done:
we'd have a subsequence $\{s_{n_k}\}$ of s_n converging to $x \in \mathbb{R}$
And then, by 7.4, $s_n \to x$ as well.
Since $\{s_n\}$ is bounded (above and below) $L = \limsup s_n$ exists and is not $\pm\infty$.
We'll show: $L \in X$ and then we're done
In fact one can show $\liminf s_n = \min X$, $L = \limsup s_n = \max X$.