Last time
- $\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$.
- $\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}$
- Notion of a Cauchy sequence

7.2 Convergent sequences are Cauchy.
- Converse is false in general. If every Cauchy sequence in a space $(E,d)$ converges, $E$ is called complete

7.3 Any Cauchy sequence is bounded.

7.4 If $\{s_n\}$ is Cauchy and a subsequence $s_{n_k} \to L$ then $s_n \to L$.

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Goal: $\mathbb{R}^n$ is complete $\forall n$.

We'll prove that $\mathbb{R}$ is complete first.

(Bolzano-Weierstrass in $\mathbb{R}$)

Lemma 8.1. Let $\{s_n\}$ be a bounded sequence in $\mathbb{R}$. $L = \limsup s_n$. Then $\exists$ a subsequence $\{s_{n_k}\}$ s.t. $s_{n_k} \to L$.

Proof Let $v_N = \sup\{s_n \mid n \geq N\}$. By definition $L = \lim_{N\to\infty} v_N$.

$\Rightarrow \forall \varepsilon > 0 \; \exists K$ s.t. $N \geq K \Rightarrow$

$$L - \varepsilon < v_N = \sup\{s_n \mid n \geq N\} < L + \varepsilon.$$

$\Rightarrow \exists i \geq N$ s.t. $L - \varepsilon < s_i \; (\leq v_N < L + \varepsilon)$

$\Rightarrow$ For $\varepsilon = 1$, $\exists K_1$ and $n_1 \geq K_1$ s.t. $L - 1 < s_{n_1} < L+1$

For $\varepsilon = 1/2$ $\exists K_2$ and $n_2 \geq K_2$ s.t. $L - 1/2 < s_{n_2} \leq v_{K_2} < L + 1/2$

Moreover by replacing $K_2$ with $\max(K_2, n_1+1)$ we may assume

$$n_2 \geq K_2 > n_1$$

Continuing this way we get a sequence

$$n_1 < n_2 < n_3 \cdots < n_k < n_{k+1} < \cdots$$

so that $L - 1/k < s_{n_k} < L + 1/k \quad \forall k$

And then $s_{n_k} \to L$.

<u>Corollary 8.2</u> $\mathbb{R}$ is complete.

<u>Proof</u> Suppose $\{s_n\} \subseteq \mathbb{R}$ is Cauchy. By 7.3 $\{s_n\}$ is bounded.

By 8.1 $\{s_n\}$ has a convergent subsequence $\{s_{n_k}\}$.

By 7.4 $s_n \to \lim s_{n_k}$. $\square$

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We have seen that $(\mathbb{R}^n, d_2)$ is a metric space where

$$d_2(x,y) = \left(\Sigma\, (x_i - y_i)^2\right)^{1/2}$$

There are two more useful metrics on $\mathbb{R}^n$

$$d_1(x,y) := \Sigma\, |x_i - y_i|$$

and $$d_\infty(x,y) := \max_{1 \le i \le n} |x_i - y_i|.$$

<u>Lemma 8.3</u> $(\mathbb{R}^n, d_1)$ is complete.

<u>Proof</u> Let $\{x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$ w.r.t. $d_1$.

<u>Note:</u> $\forall j$ $\quad |x_j - y_j| \le \sum_{i=1}^{n} |x_i - y_i| = d_1(x,y)$ (all $x, y \in \mathbb{R}^n$)

Hence $\forall \varepsilon > 0\ \exists N$ st $k, l > N \Rightarrow$

$$\varepsilon > d_1(x^{(k)}, x^{(l)}) \ge |x_j^{(k)} - x_j^{(l)}| \quad \forall j$$

$\Rightarrow \{x_1^{(k)}\}, \dots, \{x_n^{(k)}\}$ are Cauchy sequences in $\mathbb{R}$

Since $\mathbb{R}$ is complete, $\forall j$ $\exists L_j$ with $x_j^{(k)} \to L_j$.

$\Rightarrow \forall \varepsilon,\ \exists N_j$ st $k > N_j \Rightarrow |x_j^{(k)} - L_j| < \varepsilon/n$.

Then $\forall\ k > \max\{N_1, \dots, N_n\}$,

$$d_1((L_1 \dots L_n), x^{(k)}) = \sum_{j=1}^{n} |L_j - x_j^{(k)}| < \frac{\varepsilon}{n} + \frac{\varepsilon}{n} + \dots + \frac{\varepsilon}{n} = \varepsilon$$

$\therefore\ x^{(k)} \to L$ in $\mathbb{R}^n$ w.r.t. $d_1$.

<u>Question:</u> Are $(\mathbb{R}^n, d_2)$, $(\mathbb{R}^n, d_\infty)$ complete?

Definition A norm on $\mathbb{R}^n$ is a function $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$ so that

(1) $\|x\| \geq 0 \ \forall x$ and $\|x\| = 0 \iff x = \vec{0}$

(2) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall \lambda \in \mathbb{R} \quad \forall x \in \mathbb{R}^n$

(3) $\|x+y\| \leq \|x\| + \|y\|. \quad \forall x, y \in \mathbb{R}^n.$

Examples $\|x\|_2 = \left(\sum x_i^2\right)^{1/2}$ is a norm, the Euclidean norm, ($L_2$-norm

$\|x\|_1 = \sum |x_i|$ is a norm ($L_1$ - norm)

$\|x\|_\infty = \max\{|x_i|\}$ is a norm. ($L_\infty$ -norm)

Lemma 8.4 Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a norm.
Then $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, $d(x,y) = \|x-y\|$ is a metric.

Proof exercise.

Definition Two norms $\|\cdot\|$, $\|\cdot\|'$ on $\mathbb{R}^n$ are equivalent if $\exists\, c_1, c_2 > 0$ st

$$c_1 \|x\| \leq \|x\|' \leq c_2 \|x\| \qquad \forall x \in \mathbb{R}^n.$$

Two metrics $d, d'$ on a set $E$ are equivalent if $\exists\, c_1, c_2 > 0$ so that $\quad c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y) \quad \forall x, y \in E.$

Theorem 8.5

$$\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \qquad \forall x \in \mathbb{R}^n.$$

Proof Fix $x \in \mathbb{R}^n$.
$\|x\|_\infty = \max\{|x_1|, \ldots |x_n|\}$ so $\exists j$ st $\|x\|_\infty = |x_j| =$

But then $|x_j| = (x_j^2)^{1/2} \leq \left(\sum x_i^2\right)^{1/2} = \|x\|_2$

$(\|x\|_2)^2 = \sum_j |x_j|^2 \leq \left(\sum |x_j|\right)^2 = (\|x\|_1)^2$

$\therefore \|x\|_2 \leq \|x\|_1.$

$n\|x\|_\infty = \underbrace{|x_j| + \cdots + |x_j|}_{n} \geq |x_1| + |x_2| + \cdots + |x_n| = \|x\|_1.$ $\square$

<u>Remark</u> Suppose $\|\cdot\|$, $\|\cdot\|'$ are two equivalent norms:

$\exists c_1, c_2 > 0$ s.t. $c_1\|z\| \le \|z\|' \le c_2 \|z\| \quad \forall z \in \mathbb{R}^n$

Then $\forall x, y \in \mathbb{R}^n$

$$c_1 \|x-y\| \le \|x-y\|' \le c_2 \|x-y\|$$

$\Rightarrow$ the metrics $d(x,y) = \|x-y\|$ and $d'(x,y) = \|x-y\|'$ are equivalent.

<u>Lemma 8.6</u> Suppose $d, d' : E \to [0,\infty)$ are two equivalent metrics

(1) A sequence $\{s_n\} \subseteq E$ is Cauchy w.r.t $d \Leftrightarrow \{s_n\}$ is Cauchy w.r.t. $d'$

(2) A sequence $\{s_n\} \subseteq E$ converges to $L$ w.r.t $d \Leftrightarrow s_n \to L$ w.r.t $d'$.

<u>Proof</u> Since $d, d'$ are equivalent $\exists c_1, c_2 > 0$ s.t.

$$(*) \quad c_1\, d(x,y) \le d'(x,y) \le c_2\, d(x,y).$$

(1) Suppose $\{s_n\}$ is Cauchy w.r.t. $d'$.

Then given $\varepsilon > 0$ $\exists N$ s.t. $n, m \ge N \Rightarrow d'(s_n, s_m) < c_1 \varepsilon$.

Then for $n, m \ge N$

$$d(s_n, s_m) \le \frac{1}{c_1} d'(s_n, s_m) < \frac{1}{c_1} \cdot c_1 \varepsilon = \varepsilon.$$

$\therefore$ $(s_n)$ is Cauchy w.r.t to $d$.

For the converse observe that $(*)$ holds $\Leftrightarrow$

$$\frac{1}{c_2} d'(x,y) \le d(x,y) \le \frac{1}{c_1} d'(x,y).$$

(2) Suppose $s_n \to L$ w.r.t $d'$. Given $\varepsilon > 0$ $\exists N$ st

for $n \ge N$ $\quad d'(s_n, L) < c_1 \varepsilon$

Then $\quad d(s_n, L) \le \frac{1}{c_1} d'(s_n, L) < \frac{1}{c_1} c_1 \varepsilon = \varepsilon$

So $\quad s_n \longrightarrow L$ w.r.t $d$.

<u>Corollary</u> $\mathbb{R}^n$ is complete w.r.t. $d_2$ and w.r.t. $d_\infty$.