MATH 424 Eugene Lerman 02/05/2024

9.1

Last time - Proved that any Cauchy sequence in $\mathbb{R}$ converges

- Proved that any Cauchy sequence in $(\mathbb{R}^n, d_1)$ converges.

$d_1(x,y) = \sum |x_i - y_i|$

- Notion of a norm $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$.

Norms give rise to metrics: $d(x,y) = \|x-y\| \quad \forall x,y \in \mathbb{R}^n$.

Notion of equivalent norms and equivalent metrics:

d, d' are equivalent $\Leftrightarrow \exists c_1, c_2 > 0$ s.t

$$c_1 d(x,y) \le d'(x,y) \le c_2 d(x,y) \quad \forall x,y.$$

Didn't quite finish proving

Theorem 8.5

$$\frac{1}{n}\|x\|_1 \overset{①}{\le} \|x\|_\infty \overset{②}{\le} \|x\|_2 \overset{③}{\le} \|x_1\| \quad \forall x \in \mathbb{R}^n.$$

We proved ① and ②

Proof of ③: $\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \le \left(\sum |x_i|\right)^2 = \left(\|x\|_1\right)^2$.

Remark if $\|\cdot\|, \|\cdot\|'$ are two norms and $\exists c_1, c_2$ s.t

$$c_1\|x\| \le \|x\|' \le c_2\|x\| \quad \forall x$$

Then $c_1\|x-y\| \le \|x-y\|' \le c_2\|x-y\| \quad \forall x,y$

$\Rightarrow$ the associated metrics $d(x,y) = \|x-y\|$, $d'(x,y) = \|x-y\|'$ are equivalent.

Lemma 8.6 Suppose $d, d' : E \to [0,\infty)$ are two equivalent metrics. Then

(1) $\{s_n\} \subseteq E$ is Cauchy w.r.t $d \Leftrightarrow \{s_n\}$ is Cauchy w.r.t. d'

(2) $\{s_n\} \subseteq E$ converges w.r.t $d \Leftrightarrow \{s_n\}$ converges w.r.t d'.

Proof Since d, d' are equivalent $\exists c_1, c_2 > 0$ st

$$(*) \quad c_1 d(x,y) \le d'(x,y) \le c_2 d(x,y) \quad \forall x,y \in E$$

Suppose $\{s_n\}$ is Cauchy w.r.t. d'. Then given $\varepsilon > 0$

$\exists N$ st $n,m \ge N \Rightarrow d'(s_n, s_m) < c_1\varepsilon$.

and then $n, m > N \Rightarrow \quad d(s_n, s_m) < \frac{1}{c_1} d'(s_n, s_m) < \frac{1}{c_1} c_1 \varepsilon = \varepsilon.$

Conversely if $\{s_n\}$ is Cauchy w.r.t d Then

$\exists N$ s.t for $n, m > N \qquad d(s_n, s_m) < \varepsilon / c_2.$

And then $\quad d'(s_n, s_m) \le c_2 d(s_n, s_m) < c_2 \cdot \varepsilon / c_2 = \varepsilon.$

Proof of (2) is similar.

<u>Corollary</u> $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ are complete.

<u>Proof</u> Suppose $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is a metric equivalent to d_1 and $\{s_n\}$ is Cauchy in $(\mathbb{R}^n, d)$. Then $\{s_n\}$ is Cauchy in $(\mathbb{R}^n, d_1)$, hence $\{s_n\}$ converges (w.r.t. d_1)

But then $\{s_n\}$ converges w.r.t d as well.

Since d_2, d_∞ are equivalent to d_1, we're done.

<u>Recall</u> • A subset $\mathcal{O}$ of a metric space (E, d) is <u>open</u> if $\forall x \in \mathcal{O}$ $\exists r > 0$ s.t $B_r(x) \subseteq \mathcal{O}$ [and then $B_{r'}(x) \subseteq \mathcal{O}$ $\forall r' < r$]

Properties of open sets:

1) $\emptyset$, E are open
2) if $\mathcal{O}, \mathcal{O}'$ are open then so is $\mathcal{O} \cap \mathcal{O}'$
3) if $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is a collection of open sets Then $\bigcup_{\alpha \in A} \mathcal{O}_\alpha$ is open.

<u>Definition</u> A <u>topology</u> $\mathcal{T}$ on a set X is a collection of subsets of X (i.e. $\mathcal{T} \subseteq \mathcal{P}(X) :=$ the power set of X) s.t.

1) $\emptyset, X \in \mathcal{T}$
2) $\forall\ \mathcal{O}, \mathcal{O}' \in \mathcal{T}, \quad \mathcal{O} \cap \mathcal{O}' \in \mathcal{T}$
3) $\forall\ \{\mathcal{O}_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}, \quad \bigcup_{\alpha \in A} \mathcal{O}_\alpha \in \mathcal{T}$

Elements of $\mathcal{T}$ are called <u>open sets</u>.

We've seen: a metric d on a set E defines a topology, call it $\mathcal{T}_d$.

<u>Def</u> A <u>topological space</u> is a set X with a topology $\mathcal{T}$, ie a pair $(X, \mathcal{T})$.

<u>Lemma 9.1</u> Let d, d' be two equivalent metrics on a set E. Then $\underline{\mathcal{T}_d = \mathcal{T}_{d'}}$, ie d, d' define the same <u>topology</u>.

<u>Proof</u>

Enough to show: $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$ (for then, by the same argument $\mathcal{T}_{d'} \subseteq \mathcal{T}_d$)

Since d, d' are equivalent $\exists c > 0$ s.t $c\, d(x,y) \leq d'(x,y) \ \forall x, y$.

Suppose $\mathcal{O} \in \mathcal{T}_d$. Then $\forall x \in \mathcal{O}\ \exists r > 0$ s.t $B^d_r(x) \subseteq \mathcal{O}$

$y \in B^{d'}_{cr}(x) \Rightarrow cr > d'(x,y) \ (\geq c\, d(x,y))$

$\Rightarrow d(x,y) < r \Rightarrow y \in B^d_r(x)$

$\Rightarrow B^{d'}_{cr}(x) \subseteq B^d_r(x) \subseteq \mathcal{O}$

Since x is arbitrary $\mathcal{O}$ is open w.r.t d', ie $\mathcal{O} \in \mathcal{T}_{d'}$.

Since $\mathcal{O}$ is arbitrary $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$ ▷

<u>Definition</u> Let $(X, \mathcal{T})$ be a topological space, $\{s_n\} \subseteq X$ a sequence and $L \in X$. Then s_n <u>converges</u> to L if $\forall$ open set $\mathcal{U} \in \mathcal{T}$ <u>with</u> $L \in \mathcal{U}$, $\exists N$ s.t for $n > N$ $s_n \in \mathcal{U}$.

<u>Exercise</u> If $\mathcal{T} = \mathcal{T}_d$ for some metric d, the two notions of convergence agree.

<u>Corollary 9.2</u> Let E be a set, d, d' <u>two metrics</u> on E with $\mathcal{T}_d = \mathcal{T}_{d'}$. Then $s_n \to L$ in E w.r.t d $\Leftrightarrow s_n \to L$ w.r.t d'.

Compactness

Def Let $(X, \mathcal{T})$ be a topological space, $K \subseteq X$ a subset. An open cover of K is a collection $(\mathcal{O}_\alpha)_{\alpha \in A}$ of open sets s.t.
$$K \subseteq \bigcup_\alpha \mathcal{O}_\alpha.$$

Ex $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$.

Ex In any metric space (E, d)
$\{B_{1-1/n}(x)\}_{n \in \mathbb{N}}$ is an open cover of $B_1(x)$.

Definition A subset K of a topological space is compact if $\forall$ open cover $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ of K there is a finite subcover
i.e. $\alpha_1, \dots \alpha_n \in A$ s.t. $K \subseteq \mathcal{O}_{\alpha_1} \cup \dots \cup \mathcal{O}_{\alpha_n}$.

Ex Any finite set $\{x_1, \dots x_n\}$ in a top space $(X, \mathcal{T})$ is compact: if $\{x_1, \dots x_n\} \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$
Then $\forall i$, $\exists\, \mathcal{O}_{\alpha_i}$ s.t. $x_i \in \mathcal{O}_{\alpha_i}$
and then $\{x_1, \dots x_n\} \subseteq \mathcal{O}_{\alpha_1} \cup \dots \cup \mathcal{O}_{\alpha_n}$.

Ex $\mathbb{R}$ is not compact: the open cover $\{(n, n+2)\}_{n \in \mathbb{Z}}$ has no finite subcover.

Ex $\mathbb{N} \subseteq \mathbb{R}$ is not compact since
$\{(n - 1/2, n + 1/2)\}_{n \in \mathbb{N}}$ is an open cover of $\mathbb{N}$
with no finite subcover.

Theorem $[0,1] \subseteq \mathbb{R}$ (standard topology, i.e. $\mathcal{T}_d$ $d(x,y) = |x-y|$) is compact.