Review for the final exam, Math 424

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The final exam will cover lectures 1 – 38. You are expected to know definitions and be able to supply proofs of theorems that are relatively short.

Please report any issues on CAMPUSWIRE.

1 Let (X, d) be a metric space and $Y \subset X$ a subset. Define a function $d_Y : Y \times Y \to [0, \infty)$ by $d_Y(y_1, y_2) := d(y_1, y_2)$. I.e., d_Y is a restriction of d to $Y \times Y \subset X \times X$. Prove that d_Y is a metric.

2 Give an example of a connected space that is not path connected.

3 By problem 1, $(0, \infty)$ is a metric space with the metric $d(x, y) = |x - y|$.

Is the subset $(0, 1]$ of $(0, \infty)$ closed and bounded?

Is $(0, 1]$ complete?

Is $(0, 1]$ compact?

4 Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{1}{n}$ if $x \in [0, n]$ and zero otherwise.

Are these functions Lebesgue integrable on $\mathbb{R}$?

Does the sequence $\{f_n\}$ converge to an integrable function? Is convergence uniform?

Is it true that $\int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} \lim f_n \, dm$?

Does this contradict the monotone convergence theorem?

5 Prove that a *finite* subset $F \subset \mathbb{R}$ has 0 Lebesgue outer measure. Does this imply that F is measurable? Explain/prove your answer.

6 Let $f : \mathbb{R} \to [-\infty, \infty]$ be measurable and suppose $E \subset \mathbb{R}$ has measure 0. Prove that $\int_E f \, dm = 0$.

7 Suppose S is a metric space, K a compact topological space and $f : K \to S$ a continuous bijection. Prove that the inverse of f is continuous.

Give an example of a continuous bijection with no continuous inverse and prove that the inverse is not continuous.

8 Suppose $g, f : (-1, 1) \to \mathbb{R}$ are infinitely differentiable and $f^{(k)}(0) = g^{(k)}(0)$ for all k. Are f and g equal?

9 State the Cauchy criterion for Riemann integrability.

10 Prove that monotone functions are Riemann integrable.

11 Prove that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

12 Suppose $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable. Is $|f|$ Riemann integrable? Is the product fg Riemann integrable?

13 State the two versions of the fundamental theorem of calculus. Prove one version.

14 What is a complete metric space?

15 What is change of variables formula for integrals? What do you need to assume about the functions involved?

16 Compute the Taylor series of $f(x) = \ln x$ around 1.

17 Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ is a sequence of integrable functions converging to a function f. What is enough to assume to guarantee that $f = \lim f_n$ is integrable and that $\int_{[a,b} f = \lim \int_{[a,b]} f_n$?

18 Suppose $\{f_n : (a,b) \to \mathbb{R}\}$ is a sequence of differentiable functions converging to a f. Give a sufficient condition to guarantee that f is differentiable and that the sequence of derivatives f_n' converge to f'.

19 Give a sufficient condition for

$$\frac{d}{dx}\left(\int_a^b f(x,y)\,dy\right) = \int_a^b \frac{\partial}{\partial x} f(x,y)\,dy.$$

20 What does it mean for a series $\sum a_n$ to converge? To converge absolutely? To converge conditionally? Give an example of a conditionally convergent series.

21 Give an example of a function $f : [0,1] \to \mathbb{R}$ such that f is not Riemann integrable but $|f|$ is.

22 Prove that limits of sequences in a metric space are unique.

23 Let $E \subset \mathbb{R}$ be a **non-measurable** set; they do exist. Is the indicator/characteristic function χ_E measurable?

24 Let $\{I_n\}$ be a nested sequence of **closed** intervals. Is $\bigcap I_n$ non-empty? Prove your answer. Now suppose that $\{I_n\}$ be a nested sequence of **open** intervals. Is $\bigcap I_n$ non-empty? Give a proof or a counter-example.

25 Suppose $A \subset \mathbb{R}^n$ is totally bounded. Is A compact?

26 Suppose the sequence $p_n(x) = a_n x^2 + b_n x + c_n$ of polynomials converges pointwise on $[0,1]$ to a function f. Prove that f has to be a polynomial and that the sequence converges uniformly. Hint: argue first that the sequence $\{c_n\}$ converges.