Review for the first midterm exam, Math 424

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I expect you to be able to write down definitions: e.g. *least upper bound*, *subsequence*, *limit of a sequence*, *metric*, *topology*, lim sup, *interior*, *closure*... (this is not an exhaustive list).

1 Let $S \subset \mathbb{R}$ be a bounded set. Prove that there is a sequence $\{a_n\} \subset S$ that converges to $x = \inf S$.

2 Define compact, totally bounded, complete, sequentially compact. [With regard to completeness: have seen that for the real line existence of least upper bounds implies that all Cauchy sequences converge. We haven't proved the converse, but it is true.]

3 Suppose $\{x_n\}$, $\{y_n\}$ are two convergent sequence of real numbers with $x_n \leq y_n$ for all n. Prove that $\lim x_n \leq \lim y_n$.

4 Suppose $\{a_n\} \subset \mathbb{R}$ is a sequence bounded above. Is there $c \in \mathbb{R}$ with $c = \limsup a_n$? Justify your answer.

5 Consider $\mathbb{R}$ with the discrete metric d: $d(x, y) = 1$ for all $x \neq y$.
a Is the metric space $(\mathbb{R}, d)$ complete? Explain (i.e., prove your answer).
b What are the compact subsets of $(\mathbb{R}, d)$? Explain.

6 Let (S, d) be a metric space and $\{a_n\} \subset S$ is a convergent sequence. Prove that $\{a_n\}$ is Cauchy.

7 Suppose $C \subset \mathbb{R}^k$ is closed and $\{a_n\}$ is a sequence of points in C which is Cauchy. Prove that $\{a_n\}$ converges to a point in C.

8 Let (S, d) be a metric space and $\{a_n\} \subset S$ is a Cauchy sequence. Prove that $\{a_n\}$ is bounded.

9 Suppose $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are three sequence of real numbers with $x_n \leq y_n \leq z_n$ for all n. Suppose $\{x_n\}$ and $\{z_n\}$ are convergent. Is $\{y_n\}$ necessarily convergent? Explain.

10 Let (S, d) be a metric space, $C \subset S$ a closed subset and $\{a_n\} \subset S$ a sequence that converges to $L \notin C$. Prove that there is N so that $a_n \notin C$ for $n > N$.

11 Consider $\mathbb{R}$ with the standard topology. Prove that the closure of $\mathbb{Q}$ is all of $\mathbb{R}$ and that $\mathbb{Q}$ has empty interior.

You may assume that for any $x \in \mathbb{R}$ and for any $\varepsilon > 0$ the open interval $(x - \varepsilon, x + \varepsilon)$ contains a rational number and an irrational number (in fact it contains infinitely many rationals and irrationals, but you probably won't need that).

12 Prove that limits of sequences in metric spaces are unique.

13 Let (S, d) be a metric space, $\{a_n\} \subset S$ a convergent sequence and $\{a_{n_k}\}$ a subsequence. Prove that $\{a_{n_k}\}$ converges and that $\lim a_{n_k} = \lim a_n$.

14 Let $a_n = \cos(\frac{\pi n}{3})$. What are $\limsup a_n$? $\liminf a_n$?

15 Suppose (E, d) is a metric space and $A \subseteq E$ is closed and bounded. Is A necessarily compact? Explain.

16 Suppose (E, d) is a sequentially compact metric space. Is E totally bounded? Explain.

17 Suppose $A \subset \mathbb{R}^n$ is totally bounded ($\mathbb{R}^n$ is given the standard topology). Is A necessarily compact?

18 Suppose $f : X \to Y$ is a continuous map between two topological spaces and $K \subseteq X$ is compact. Prove that the image $f(K)$ is compact in Y.

19 Suppose $f : X \to Y$ and $g : Y \to Z$ are two continuous maps between topological spaces. Prove that $g \circ f : X \to Z$ is continuous.

20 Suppose $f : X \to Y$ is a continuous map and $C \subset Y$ is closed. Prove that the preimage $f^{-1}(C)$ is closed in X. [Here is an "application": for any topological space X and any continuous function $f : X \to \mathbb{R}$ the level sets $f^{-1}(c) = \{x \in X \mid f(x) = c\}$ are closed for all $c \in \mathbb{R}$. Why does this follow?]

21 Consider the map $f : [0, 2\pi) \to S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$; it's a continuous bijection. Prove that the inverse $f^{-1} : S^1 \to [0, 2\pi)$ cannot be continuous.

Hint: Prove that S^1 is closed and bounded; problem **20— should be useful.**