Review for the second midterm exam, Math 424

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The second midterm will cover lectures 15 – 29. You are expected to know definitions and be able to supply proofs of theorems that are relatively short.

Please report any issues on CAMPUSWIRE.

1 Suppose X is a connected topological space and $f : X \to Y$ is continuous. Prove that the image $f(X)$ is connected.

2 Prove that any path connected space is connected.

Hint: you may (and you would need to) assume that $[0, 1]$ is connected.

3 Is every connected space path connected? Is a connected open subset of $\mathbb{R}^n$ path connected? Explain.

4(a) State and prove Rolle's theorem. You may use the fact that at extremal points of a differentiable function its derivative vanishes.

4(b) Use Rolle's theorem to prove the Mean Value theorem.

5 Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$. Is f continuous at x_0? Can you prove your answer?

6 What does the inverse function theorem say?

7 What does Taylor's theorem say?

8 Suppose $f : (-1, 1) \to \mathbb{R}$ is infinitely differentiable and all the derivatives of f at 0 are zero: $f^{(k)}(0) = 0$ for all k. Does f have to be constant? What's the Taylor series of f?

9 State the Cauchy criterion for integrability.

10 Prove that monotone functions are integrable.

11 Prove that continuous functions are integrable.

12 Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable. Is $|f|$ integrable? Is the product fg integrable?

13 State the two versions of the fundamental theorem of calculus.

14 State and prove integration by parts.

15 What is change of variables formula for integrals? What do you need to assume about the functions involved?

16 Define natural logarithm function $\ln(x)$ in terms of an integral.

17 Suppose $\{f_n : [a, b] \to \mathbb{R}\}$ is a sequence of integrable functions converging to a function f. What is enough to assume to guarantee that $f = \lim f_n$ is integrable and that $\int_{[a,b} f = \lim \int_{[a,b]} f_n$?

18 Suppose $\{f_n : (a,b) \to \mathbb{R}\}$ is a sequence of differentiable functions converging to a f. Give a sufficient condition to guarantee that f is differentiable and that the sequence of derivatives f_n' converge to f'.

19 Give a sufficient condition for

$$\frac{d}{dx}\left(\int_a^b f(x,y)\,dy\right) = \int_a^b \frac{\partial}{\partial x} f(x,y)\,dy.$$

20 What does it mean for a series $\sum a_n$ to converge? To converge absolutely? To converge conditionally? Give an example of a conditionally convergent series.

21 Give an example of a function $f : [0,1] \to \mathbb{R}$ such that f is not integrable but $|f|$ is. If f is integrable, is $|f|$ necessarily integrable?

22 Prove the mean value theorem for integrals: if $f : [a,b] \to \mathbb{R}$ is continuous, then there is $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

23 Prove or give a counterexample: if f, g are integrable on $[a,b]$ and h is a function such that

$$f(x) \leq h(x) \leq g(x)$$

for all $x \in [a,b]$ then h is integrable on $[a,b]$.

24 Let $F(x) = \int_0^x xe^{t^2}\,dt$. Find $F''(x)$ for $x \in (0,1)$. Caution: $F'(x) \neq xe^{x^2}$.

25 Prove that if $\sum |a_n|$ converges and $\{b_n\}$ is a bounded sequence then $\sum a_n b_n$ converges. [This is a lot easier to prove than Dirichlet criterion.]

26 Let $S \subset \mathbb{R}$ be a set bounded above and $c > 0$. Prove that

$$\sup\{cx \mid x \in S\} = c\sup\{x \mid x \in S\}.$$