Homework #11 Math 518
Due in class Wednesday, November 9, 2011

1. In the previous homework you have defined the exponential map

$$\exp : \mathfrak{g} \to G$$

for any Lie group G. Prove that for any $X \in \mathfrak{g}$ the curve $\sigma(t) = \exp(tX)$ is the integral curve of the left invariant vector field $\tilde{X}$ defined by X. Hint: show that if $\gamma(\tau)$ is an integral curve of the vector field Y then for any $t \in \mathbb{R}$ the curve $\tau \mapsto \gamma(t\tau)$ is an integral curve of the vector field tY.

2. Prove that the exponential map is natural. That is, given a map of Lie groups $f : G \to H$, show that for any $X \in \mathfrak{g}$ we have

$$\exp(\delta f(X)) = f(\exp(X)),$$

where exp on the left denotes the exponential map for the group H, exp on the right denotes the exponential map for G and $\delta f : \mathfrak{g} \to \mathfrak{h}$ is the induced map of Lie algebras, i.e., $\delta f = df_1$.

3 Recall that the $n-1$ sphere S^{n-1} can be thought of as $\{v \in \mathbb{R}^n \mid ||v||^2 = 1\}$. Consider

$$L := \{(v, w) \in S^{n-1} \times \mathbb{R}^n \mid w = \lambda v \text{ for some } \lambda \in \mathbb{R}\}.$$

Prove that L is a trivial vector bundle over S^{n-1} of rank 1 (you may assume that L is a manifold).

4 Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles over M.
(a) Show that $E \times F$ is a vector bundle over $M \times M$.
(b) Explain why the fiber product

$$G = E \times_M F = \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f)\}$$

can be considered a vector bundle over M. Hint: consider the diagonal map $\Delta : M \to M \times M$...
(c) Show that, as a vector bundle over M, G is isomorphic to $E \oplus F$.

5 Consider the 2-form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

on $\mathbb{R}^4$. Compute $\omega \wedge \omega \in \Omega^4(\mathbb{R}^4)$.