Homework #12 Math 518
Due in class Wednesday, November 16, 2011

1. Let M be an open subset of $\mathbb{R}^3$. In $\mathbb{R}^3$, the standard inner product $(\cdot,\cdot)$ defines an isomorphism $\mathbb{R}^3 \to (\mathbb{R}^3)^*$, $v \mapsto (v,\cdot)$, which in turn induces an isomorphism of spaces of sections

$$A : \Gamma(TM) \to \Omega^1(M), \quad A(X) = (X,\cdot).$$

The standard volume form $\mu = dx_1 \wedge dx_2 \wedge dx_3$ defines an isomorphism $\mathbb{R}^3 \to \Lambda^2((\mathbb{R}^3)^*)$ by $v \mapsto \iota(v)\mu$, which also induces an isomorphism

$$B : \Gamma(TM) \mapsto \Omega^2(M) \quad B(X) = \iota(X)\mu.$$

Finally, the map

$$C : C^\infty(M) \to \Omega^3(M) \quad C(f) = f\mu$$

is also an isomorphism. (Check these facts!)

Show that the standard vector calculus notions of div, grad, and curl can be defined as

1. $\mathrm{grad}(f) = A^{-1}(df)$ for any smooth function f on M.
2. $\mathrm{curl}(X) = B^{-1}(d(A(X)))$ for any vector field X on M.
3. $\mathrm{div}(X) = C^{-1}(d(B(X)))$ for any vector field X on M.

In other words prove that the diagram

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "grad"] \arrow[d, "="'] & \Gamma(TM) \arrow[r, "curl"] \arrow[d, "A"] & \Gamma(TM) \arrow[r, "div"] \arrow[d, "B"] & C^\infty(M) \arrow[d, "C"] \\
C^\infty(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M)
\end{tikzcd}$$

commutes.

2. Prove that for any manifold M, a 1-form α on M and any vector fields X, Y on M,

$$L_X(\imath(Y)\alpha) = \imath(L_X Y)\alpha + \imath(Y)L_X\alpha.$$

3 Let V be a real vector space of dimension n, $v_1, \dots, v_k \in V$ a finite collection of vectors. Prove:

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$$

if and only if the set $\{v_1, \dots, v_k\}$ is linearly independent.

4 Prove that for any k-form α and any ℓ-form β on a manifold M, we have

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha.$$