Homework #13 Math 518 (version 2, definition of inner product corrected 11/27)
Due in class Wednesday, November 30, 2011

Recall that an inner product on a (finite dimensional) real vector space V is a bilinear map $g : V \times V \to \mathbb{R}$ (i.e., $g \in V^* \otimes V^*$) which is

(i) symmetric: $g(v, w) = g(w, v)$ for all $v, w \in V$; and

(ii) positive definite: $g(v, v) > 0$ for all **nonzero** vectors $v \in V$.

1. Prove that an inner product g on a vector space V defines an **isomorphism** $g^\# : V \to V^*$ with

$$(g^\#(v))(w) := g(v, w)$$

for all $v, w \in V$. That is, $g^\#(v) = g(v, -) \in V^*$.

2. A *Riemannian metric* g on a manifold M is a smooth section of the bundle $T^*M \otimes T^*M \to M$ so that for all $q \in M$, the bilinear map $g_q : T_qM \times T_qM \to \mathbb{R}$ is an inner product. If g is a Riemannian metric on a manifold M, the pair (M, g) is called a Riemannian manifold. Prove that every manifold M has a Riemannian metric. Hint: Construct the metric locally in coordinates first.

3. Prove that if (M, g) is a Riemannian manifold then the metric g defines an isomorphism of vector bundles $g^\# : TM \to T^*M$ by

$$T_qM \ni v \mapsto g_q(v, -) \in T_q^*M$$

for all $(q, v) \in T_qM$. That is

$$g^\#(q, v) = g_q(v, -).$$

4. Prove that given a smooth function f on a Riemannian manifold (M, g) there is a unique vector field ∇f with

$$g_q(\nabla f_q, v) = df_q(v)$$

for all $q \in M$, $v \in T_qM$. This vector field ∇f is called the *gradient vector field* of f.

5 Let f be a smooth function on a Riemannian manifold (M, g) and $\gamma(t)$ an integral curve of the gradient vector field ∇f of f. Prove that

$$\frac{d}{dt} f(\gamma(t)) \geq 0$$

for all t that γ is defined. When is $\frac{d}{dt} f(\gamma(t)) = 0$?