Homework #4 Math 518
Due in class Wednesday, September 21, 2011

1. Prove that if $F : M \to N$ is a bijective map of manifolds such that the differential dF_x is an isomorphism for all points $x \in M$ then F is a diffeomorphism.

2(a). Prove that for any manifold M the diagonal

$$\Delta_M := \{(x, y) \in M \times M \mid x = y\}$$

is an embedded submanifold.

2(b). Show that the map $\delta : M \to \Delta_M$, $\delta(x) := (x, x)$ is a diffeomorphism.

3. Suppose that $F : N_1 \to M$, $G : N_2 \to M$ are two maps of manifolds with G being a *submersion* (this means that for any point $x \in N_2$ the differential $dG_x : T_x N_2 \to T_{G(x)} M$ is onto). Prove that the *fiber product*

$$N_1 \times_M N_2 := \{(x_1, x_2) \in N_1 \times N_2 \mid F(x_1) = G(x_2)\}$$

is an embedded submanifold of $N_1 \times N_2$. Hint: consider $F \times G : N_1 \times N_2 \to M \times M$, $(F \times G)(x_1, x_2) := (F(x_1), G(x_2))$. Show that this map is transverse to Δ_M.

4. Suppose that $F : N \to M$, $G : N \to M$ are two maps with G being a submersion. Is it true that the set

$$Q := \{x \in N \mid F(x) = G(x)\}$$

is a submanifold of N? Give an example showing that if neither F nor G are submersions then Q need not be a manifold.[1]

[1] This problem may well be much harder than problem 3. Do your best.