Homework #6 Math 518
Due in class Wednesday, October 5, 2011

1. (a) Show that if $i : Q \hookrightarrow M$ is an embedded submanifold, then for any $q \in Q$ the differential $di_q : T_qQ \to T_qM$ is injective, hence we have a natural inclusion $di : TQ \hookrightarrow TM$.
(b) Show that if $Q \hookrightarrow M$ is an embedded submanifold, then $TQ \hookrightarrow TM$ is also an embedded submanifold. What are the coordinates on TM adapted to TQ?

2. Compute the flow of the vector field $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

3. Suppose a vector field X on a manifold M is zero at a point $p \in M$. Show that there is a neighborhood U of p in M so that for each point $q \in U$ the integral curve through q exists for all $t \in [-1, 1]$. Hint: this is really an exercise in understanding product topology on $\mathbb{R} \times M$. Tube lemma may help too.

4. The point of this problem is to show that nothing interesting happens to the flow of a vector field away from the points where the vector field is 0. You may be able to find a solution to this problem in any number of books. If you do, read it, understand it and write up the solution in your own words.

Let X be a vector field on a manifold M. Suppose $X_q \neq 0$. Show that there is a coordinate chart $\phi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ with $q \in U$ so that

$$X_p = \frac{\partial}{\partial x_1}\Big|_p \quad \text{for all } p \in U.$$

Hints: (i) Show first that there is a coordinate chart $\psi = (y_1, \ldots, y_m) : V \to \mathbb{R}^m$ so that $\psi(q) = 0$ and

$$X_q = \frac{\partial}{\partial y_1}\Big|_q$$

(just one point q, not all the points in V).

(ii) We know that there is $\epsilon > 0$ so that flow Φ of X exists for time $t \in (-\epsilon, \epsilon)$ for all $p \in V$. Define the map

$$f : (-\epsilon, \epsilon) \times (V \cap (\{0\} \times \mathbb{R}^{m-1})) \to M$$

by

$$f(t, (0, r)) = \Phi(t, \psi^{-1}(0, r)).$$

Compute the differential $df_{(0,(0,0))}$ and show that it is onto.