Homework #7 Math 518
Due in class Wednesday, October 12, 2011

1. Let M be a manifold. An **isotopy** on M is a collection of diffeomorphisms $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ such that

1. f_0 is the identity, and
2. the map $(-\epsilon, \epsilon) \times M \to M$ given by $(t, m) \mapsto f_t(m)$ is smooth.

A **time-dependent vector field** $\{X_t\}$ is a smooth map $(-\epsilon, \epsilon) \times M \to TM$ of the form $(t, m) \mapsto X_t(m) \in T_mM$. An isotopy $\{f_t\}$ defines a time-dependent vector field $\{X_t\}$ by

$$X_s(f_s(m)) = \frac{d}{dt}\Big|_{t=s} f_t(m).$$

Prove that given a time-dependent vector field $\{X_t\}$, there is an isotopy $\{f_t\}$ such that the equation above holds.
Hint: Let $\overline{X}(t, m) = (\frac{d}{dt}, X_t(m))$; it is a vector field on $\mathbb{R} \times M$. The local flow $\Phi_s(t, m)$ of $\overline{X}$ is of the form $\Phi_s(t, m) = (\Phi^1_s(t, m), \Phi^2_s(t, m))$. Show that $\Phi^1_s(t, m) = s + t$.

2. Consider a time-dependent vector field $X_t(m) = t\frac{d}{d\theta}$ on S^1. Compute the corresponding isotopy.

3. Suppose that M and N are manifolds. If $X \in \Gamma(TM)$ is a vector field, show that $\bar{X} : M \times N \to T(M \times N) \simeq TM \times TN$ given by $\bar{X}(m, n) = (X(m), 0)$ is a well-defined vector field on $M \times N$. Similarly, given $Y \in \Gamma(TN)$ we get $\bar{Y} \in \Gamma(T(M \times N))$. Show that $[\bar{X}, \bar{Y}] = 0$.

4. Show that if $\{v_i\}$ is a basis of a vector space V, $\{v^*_i\}$ the dual basis of V^* and $\{w_j\}$ a basis of a vector space W, then $\{v^*_i(\cdot)w_j\}$ is a basis of the vector space $\mathrm{Hom}(V, W)$ of linear maps from V to W.