Multilinear maps and tensors

Definition 19.1. Let $V_1, \dots, V_n$ and U be finite dimensional vector spaces. Then a map $f : V_1 \times \cdots \times V_n \to U$ is *multilinear* if it is linear in each variable. More concretely, for each index i and a fixed $(n-1)$-tuple $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$, the map $v \mapsto f(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ is linear.

Remark 19.2. A multilinear map $f : V_1 \times V_2 \to U$ is called a *bilinear* map.

Remark 19.3. In this class we will mostly consider vector spaces over $\mathbb{R}$. Occasionally we will consider vector spaces over $\mathbb{C}$.

Example 19.4. The map $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ such that

$$(v_1, \dots, v_n) \mapsto \det(v_1 | \cdots | v_n)$$

is multilinear. Here on the right we think of the v_is as column vectors of the matrix $(v_1 | \cdots | v_n)$.

Example 19.5. The cross product $\times : \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \to \mathbb{R}^3$ is bilinear.

Example 19.6. If $\mathfrak{g}$ is a Lie algebra, then the Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.

Notation. $\mathrm{Mult}(V_1, \dots, V_n; U) = \{f : V_1 \times \cdots \times V_n \to U \mid f \text{ is multilinear}\}$

Remark 19.7. $\mathrm{Mult}(V_1, \dots, V_n; U)$ is a vector space.

Lemma 19.8. *Suppose that V, W, U are finite dimensional vector spaces over $\mathbb{R}$. Let V have basis $\{v_i\}$ and dual basis $\{v_i^*\}$. Let W have basis $\{w_j\}$ and dual basis $\{w_j^*\}$. Let U have basis $\{u_k\}$ and dual basis $\{u_k^*\}$. Then $\{{\phi_{ij}}^k : V \times W \to U \mid {\phi_{ij}}^k(v,w) = v_i^*(v)\, w_j^*(w)\, u_k\}$ is a basis of* $\mathrm{Mult}(V, W; U)$.

Proof. If $b : V \times W \to U$ is bilinear, then for all $v \in V$ and all $w \in W$:

$$\begin{aligned} b(v,w) &= b\Big(\sum v_i^*(v)\, v_i,\ \sum w_j^*(w)\, w_j\Big) \\ &= \sum v_i^*(v)\, w_j^*(w)\, b(v_i, w_j) \\ &= \sum v_i^*(v)\, w_j^*(w)\, u_k^*\big(b(v_i, w_j)\big) u_k \\ &= \sum u_k^*\big(b(v_i, w_j)\big)\, {\phi_{ij}}^k(v,w) \end{aligned}$$

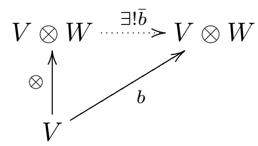
Moreover, $\{{\phi_{ij}}^k\}$ are linearly independent. Suppose not, then for all v_r, w_s in the basis:

$$\begin{aligned} 0 &= \sum {c^{ij}}_k\, {\phi_{ij}}^k \\ &= \sum {c^{ij}}_k\, v_i^*(v_r)\, w_j^*(w_s)\, u_k \\ &= \sum {c^{rs}}_k\, u_k \end{aligned}$$

Since $\{u_k\}$ is a basis, ${c^{rs}}_k = 0$ for all r, s, k. ∎

Corollary 19.9. $\dim \mathrm{Mult}(V, W; U) = \dim V \dim W \dim U$

Definition 19.10. The tensor product of two vector spaces V and W is another vector space $V \otimes W$ together with a bilinear map $\otimes : V \times W \to V \otimes W$ such that for *any* bilinear map $b : V \times W \to U$ there exists a unique $\bar{b} : V \otimes W \to U$ with $b = \bar{b} \circ \otimes$. In other words the diagram



commutes.

Notation. $\otimes(v,w)$ is usually written $v \otimes w$. Hence $b(v,w) = \bar{b}(v \otimes w)$ for all $v \in V$ and all $w \in W$.

Lemma 19.11. *$V \otimes W$ exists. (i.e. $V \times W \xrightarrow{\otimes} V \otimes W$ exists)*

Lemma 19.12. *$V \otimes W$ is unique up to isomorphism.*

Remark 19.13. Given a set X, there exists a vector space $F(X)$ with basis X. Informally $F(X)$ is the vector space of all finite linear combinations of elements of X:

$$F(X)\left\{\sum_{i=1}^{n} a_i x_i \;\middle|\; a_i \in \mathbb{R}, x_i \in X\right\}.$$

Formally we can define $F(X)$ as a certain subspace of the space of all real-valued functions on X:

$$F(X) = \{f : X \to \mathbb{R} \mid f(x) = 0 \text{ for all but finitely many } x \in X\}$$

In this case we have an "inclusion" $X \hookrightarrow F(X)$ given by $x \mapsto \delta_x$ where $\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

The inclusion $X \hookrightarrow FX$ has the following universal property:
For any vector space U and any map of sets $\varphi : X \to U$, there exists a unique linear map $\tilde{\varphi} : F(X) \to U$ such that $\tilde{\varphi}\big|_X = \varphi$.

Proof of Lemma 19.11. We can construct $V \otimes W$ as a quotient of $F(V \times W)$:

$$V \otimes W = F(V \times W)/K$$

Note that by the universal property of F we have a canonical map $V \times W \hookrightarrow F(V \times W)$. consider the following collection S of vectors in $F(V \times W)$:

$$S = \left\{ \begin{array}{c} (v_1 + v_2, w) - (v_1, w) - (v_2, w) \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ c(v, w) - (cv, w) \\ c(v, w) - (v, cw), \end{array} \;\middle|\; v, v_1, v_2 \in V,\ w, w_1, w_2 \in W \text{ and } c \in \mathbb{R} \right\}$$

and set

$$K \overset{\text{def}}{=} \operatorname{span}_{\mathbb{R}} S.$$

Now define

$$V \otimes W \overset{\text{def}}{=} F(V \times W)/K$$

and define $\otimes : V \times W \to V \otimes W$ to be the following composite map:

$$V \times W \longrightarrow F(V \times W) \longrightarrow F(V \times W)/K$$

Then $\otimes$ is bilinear by construction.
It remains to check the universal Property of $\otimes : V \times W \to V \otimes W$. Suppose that $b : V \times W \to U$ is bilinear. By the universal property of $V \times W \hookrightarrow F(V \times W)$ there exists a linear map $\tilde{b} : F(V \times W) \to U$ such that $\tilde{b}\big|_{V \times W} = b$. Since b is bilinear, for all $v_1, v_2 \in V$ and all $w \in W$ we have

$$\tilde{b}\big((v_1 + v_2, w) - (v_1, w) - (v_2, w)\big) = b(v_1 + v_2, w) - b(v_1, w) - b(v_2, w) = 0$$

Similarly

$$\tilde{b}\big((v, w_1 + w_2) - (v, w_1) - (v, w_2)\big) = 0$$

for all $v \in V$, $w_1, w_2 \in W$ and

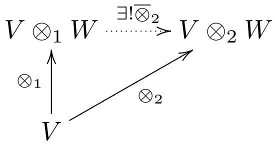
$$\tilde{b}\big(c(v, w) - (cv, w)\big) = 0 = \tilde{b}\big(c(v, w) - (v, cw)\big)$$

for all $c \in \mathbb{R}$, $v \in V$, $w \in W$. These computations imply that $\tilde{b}\big|_S = 0$. In turn this implies that $\tilde{b}\big|_K = 0$. Therefore there exists a unique linear map $\bar{b} : V \otimes W \to U$ with $\bar{b}(V \otimes W) = b(V, W)$. $\square$

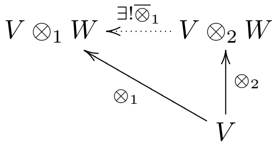
Proof of Lemma 19.12. Suppose that we have two bilinear maps:

$$\otimes_1 : V \times W \to V \otimes_1 W$$
$$\otimes_2 : V \times W \to V \otimes_2 W$$

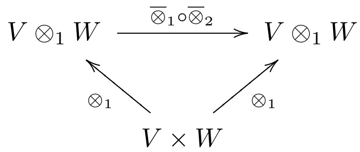
with the universal property. Then by the universal property of $\otimes_1$, there exists a unique linear map $\overline{\otimes}_2 : V \otimes_1 W \to V \otimes_2 W$ so that



commutes Similarly, by the universal property of $\otimes_2$ there exists a unique linear map $\overline{\otimes}_1 : V \otimes_2 W \to V \otimes_1 W$ so that



commutes. Therefore the following diagram commutes



By uniqueness, $\overline{\otimes}_1 \circ \overline{\otimes}_2 = \mathrm{id}_{V \otimes_1 W}$. Similarly $\overline{\otimes}_2 \circ \overline{\otimes}_1 = \mathrm{id}_{V \otimes_2 W}$. So $\overline{\otimes}_1$ and $\overline{\otimes}_2$ are the desired isomorphism. □

Lemma 19.14. *The map* $\varphi : \mathrm{Hom}(V \otimes W, U) \to \mathrm{Mult}(V \times W, U)$, $\varphi(A) = A \circ \otimes$ *is an isomorphism of vector spaces.*

Proof. By the universal property of $\otimes$ for any $b \in \mathrm{Mult}(V \times W, U)$ there is a unique $\bar{b} \in \mathrm{Hom}(V \otimes W, U)$ with $b = \bar{b} \circ \otimes$. Hence φ is one-to-one and onto. □

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