Last Time. We proved (Corollary 19.8) that $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Corollary 20.1 (6.7.1 in online notes)**.** *If V is a vector space with basis $\{v_i\}_{i=1}^n$ and W is a vector space with basis $\{w_j\}_{j=1}^m$, then $\{v_i \otimes w_j\}_{i,j}$ is a basis of $V \otimes W$.*

Proof. We know that $\{v \otimes w \big| v \in V, w \in W\}$ spans $V \otimes W$ by our construction of $V \otimes W$. In particular, $\{v_i \otimes w_j\}_{i,j}$ spans because for any $v \in V$ $w \in W$ we have

$$v \otimes w = \Big(\sum v_i^*(v)\, v_i\Big) \otimes \Big(\sum w_j^*(w)\, w_j\Big) = \sum v_i^*(v)\, w_j^*(w)\, v_i \otimes w_j.$$

□

Lemma 20.2 (6.8 in online notes)**.** *The map $V \times W \to W \times V$, $(v, w) \mapsto (w, v)$ induces a linear isomorphism $\varphi : V \otimes W \to W \otimes V$, $\varphi(v \otimes w) = w \otimes v$ for all $v \in V$ and all $w \in W$.*

Proof. The map $b : V \times W \to W \otimes V$, $b(v, w) = w \otimes v$ is bilinear. Therefore by the universal property there exists a unique map $\varphi : V \otimes W \to W \otimes V$, $\varphi(v \otimes w) = w \otimes v$. Moreover, the map is surjective because $\{w \otimes v \big| w \in W, v \in V\}$ spans $W \otimes V$. By dimension count, φ is an isomorphism. □

Lemma 20.3 (6.9 in online notes)**.** *There is a natural isomorphism $\varphi : V^* \otimes W \to \mathrm{Hom}(V, W)$ of vector spaces such that $\varphi(l \otimes w)(v) = l(v)\, w$.*

Proof. Consider the bilinear map $b : V^* \times W \to \mathrm{Hom}(V, W)$ given by $b(l, w) = l(-)\, w$ for all $l \in V^*$, $w \in W$. By the universal property of $V^* \otimes W$ there exists a unique linear map $\varphi : V^* \otimes W \to \mathrm{Hom}(V, W)$ with $\varphi(l \otimes w) = l(-)\, w$. Moreover, if $\{v_i\}$ is a basis of V, $\{v_i^*\}$ is a dual basis of V^*, and $\{w_j\}$ is a basis of W, $\{w_j^*\}$, then $\{v_i^* \otimes w_j\}$ is a basis of $V^* \otimes W$ and $\{v_i^*(-)\, w_j\}$ is a basis of $\mathrm{Hom}(V, W)$ (prove this!). Hence φ is an isomorphism. □

Lemma 20.4 (6.10 in online notes)**.** *A pair of linear maps $A : V \to W$, $A' : V' \to W'$ determine a unique linear map $A \otimes A' : V \otimes V' \to W \otimes W'$, $(A \otimes A')(v \otimes v') = Av \otimes A'v'$.*

Proof. The map $V \times V' \to W \otimes W'$, $(v, v') \mapsto Av \otimes A'v'$ is bilinear. □

Exercise 20.1. The map $\mathbb{R} \otimes V \to V$, $r \otimes v \mapsto rv$ is a well-defined isomorphism.

Exercise 20.2. For all vector spaces U, V, W there exists an isomorphism $U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ such that $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$.

Notation.

$$\begin{aligned} V^{\otimes 0} &:= \mathbb{R} \\ V^{\otimes 1} &:= V \\ V^{\otimes 2} &:= V \otimes V \\ &\vdots \\ V^{\otimes n} &:= V^{\otimes n-1} \otimes V \end{aligned}$$

Exercise 20.3. For all vector spaces V, U there exists a natural isomorphism $\mathrm{Hom}(V^{\otimes n}, U) \to \mathrm{Mult}(\overbrace{V, \dots, V}^{n}; U)$.

Definition 20.5. An *associative algebra* A is a vector space A over $\mathbb{R}$ together with an $\mathbb{R}$-bilinear map $m : A \times A \to A$, $m(a, b) = ab$ such that $m\big(m(a, b), c\big) = m\big(a, m(b, c)\big)$ (that is, $(ab)c = a(bc)$).

Definition 20.6 (6.16 in online notes)**.** An associative algebra is *graded* (by nonnegative integers) if

$$A = \bigoplus_{i=0}^{\infty} A_i = \sum_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

and if for all $a \in A_i$ and all $b \in A_j$, then $ab \in A_{i+j}$.

Example 20.7. Consider the algebra $\mathbb{R}[x]$ of polynomials in one variable. It is graded by degrees of monomials: $\mathbb{R}[x] = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}x^2 \oplus \cdots$.

Definition 20.8 (after 6.16 in online notes)**.** The *tensor algebra* of a vector space V is an associative algebra $\mathcal{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ with graded multiplication $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(n+m)}$ given by $(a, b) \mapsto a \otimes b$.

Remark 20.9. Notice that $\mathcal{T}(V)$ has a unit $1 \in \mathbb{R}$.

Definition 20.10 (6.19 in online notes)**.** Let V be a finite dimensional vector space over $\mathbb{R}$. A *Grassmann algebra* $\Lambda^* V$ on the vector space V is a graded associative algebra over $\mathbb{R}$ with unit together with an injective linear map $i : V \hookrightarrow \Lambda^* V$, $\Lambda^1 V = i(V)$ and the following universal property: For any associative algebra A and any linear map $j : V \to A$ such that $j(v)\, j(w) = 0$ there exists a unique map of algebras $\bar{j} : \Lambda^* V \to A$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda^* V \arrow[r, dotted, "\bar{j}"] & A \\ V \arrow[u, hook, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Typeset by R. S. Kueffner II