Last Time. We defined the *Grassmann (exterior) algebra* Λ^*V on a finite dimensional vector space V as an (associative) algebra (with unit) with an injective linear map $i : V \hookrightarrow \Lambda^*V$ and the universal property that for any associative algebra A and any linear map $j : V \to A$ such that $j(v)\cdot j(w) = 0$, there exists a unique map of algebras $\bar{j} : \Lambda^*V \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\Lambda^*V \arrow[r, dotted, "\bar{j}"] & \Lambda^*W \\
V \arrow[u, hook, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Remark 21.1. If A is an algebra, $V \subseteq A$ a subspace, and for all $v \in V$ $v^2 = 0$, then for all $v, w \in V$:

$$0 = (v + w)^2 = v\cdot v + v\cdot w + w\cdot v + w\cdot w = v\cdot w + w\cdot v$$

and therefore $w\cdot v = -v\cdot w$ for all $v, w \in V$.

Proposition 21.2 (6.20 in online notes)**.** *If the Grassmann algebra $i : V \hookrightarrow \Lambda^*V$ exists, then it is unique up to a (unique) isomorphism.*

Proof. Similar to the respective proof for tensors. $\square$

Remark 21.3.

(1) If A is an associative algebra, then a *two-sided ideal* $I \subseteq A$ is a linear subalgebra such that for all $a, b \in A$ and all $x \in I$, both $ax \in I$ and $xb \in I$.
(2) If $I \subseteq A$ is an ideal, then A/I is an algebra and $\pi : A \to A/I$, $a \mapsto a + I$ is a map of algebras.
(3) For any subset $S \subseteq A$ there exists a smallest ideal $\langle S \rangle$ containing S. Concretely

$$\langle S \rangle = \left\{ \sum a_i\, s_i\, b_i \;\middle|\; a_i, b_i \in A, s_i \in S \right\}$$

Proposition 21.4 (6.21 in online notes)**.** *The exterior algebra $i : V \hookrightarrow \Lambda^*V$ exists.*

Proof. Let $A = \mathcal{T}(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots$ and let $I = \langle \{ v \otimes v \mid v \in V \} \rangle$, the two-sided ideal in $\mathcal{T}(V)$ generated by squares of vectors. Note that $I = \bigoplus_n (I \cap V^{\otimes n})$. Moreover, $I \cap V^{\otimes 0} = 0$ and $I \cap V^{\otimes 1} = 0$ (by degree count: the squares $v\otimes v$ have degree 2, so any element of I has degree 2 or bigger). Let $\Lambda^*V = \mathcal{T}(V)/I$. Then $\mathbb{R} \oplus V = V^{\otimes 0} \oplus V^{\otimes 1} \hookrightarrow \mathcal{T}(V) \xrightarrow{\pi} \Lambda^*V$ is injective. Also, $\Lambda^*V = \bigoplus_{n=0}^{\infty}$ $Ext[n]V$ with $\Lambda^n V \overset{\text{def}}{=} V^{\otimes n}/(I \cap V^{\otimes n})$. Consequently $\Lambda^0 V \simeq \mathbb{R}$ and $\Lambda^1 V \simeq V$. So set $i : V \to \Lambda^*V$ to be the isomorphism $V \xrightarrow{\sim} \Lambda^1 V$.
We need to check that the constructed map $i : V \hookrightarrow \Lambda^*V$ has the desired universal property. To this end suppose that we have a linear map $j : V \to A$, where A is some associative algebra, with $j(v)\cdot j(v) = 0$ for all $v \in V$. Then for each $n \geq 1$ we have a map $V \times \cdots \times V \to A$, which is given by $(v_1, \ldots, v_n) \mapsto j(v_1) \cdots j(v_n)$. This map is n-linear. Therefore (for each n) we get a linear map $j^{(n)} : V^{\otimes n} \to A$ with $j^{(n)}(v_1 \otimes \cdots \otimes v_n) = j(v_1) \cdots j(v_n)$. Putting the maps $j^{(n)}$ together we get one map of algebras $\tilde{j} : \mathcal{T}(V) \to A$. Since $j(v)\cdot j(v) = 0$ for all $v \in V^{\otimes 1}$ it follows that $j(v \otimes v) = 0$ for all $v \in V^{\otimes 1}$. This implies that $\tilde{j}\big|_{\langle \{ v\otimes v \mid v \in V \} \rangle} \equiv 0$. Hence $\tilde{j}$ descends to $\bar{j} : \Lambda^*V = \mathcal{T}(V)/I \to A$ with $\bar{j}(v) = j(v)$ for all $v \in \Lambda^1 V$. $\square$

Notation. The product in Λ^*V is denoted by $\wedge$. So $\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$. By construction $v \wedge v = 0$ for all $v \in V$. Hence $w \wedge v = -v \wedge w$ for all $v, w \in V$.

Remark 21.5. If $\{v_1, \ldots, v_n\}$ is a basis of V then $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{(i_1 \cdots i_k) \in \{1,\ldots,n\}^k}$ is a basis of $V^{\otimes k}$. Then $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}_{(i_1 \cdots i_k) \in \{1,\ldots,n\}^k}$ generates $\Lambda^k V$. Because $\wedge$ is alternating we may assume $i_1 < \cdots < i_k$ and the resulting smaller set $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}_{i_1 < \cdots < i_k}$ still generates $\Lambda^k V$. Furthermore, if $k > n$ then some i_j's in the indexing k-tuple must repeat. Consequently $\Lambda^k V = 0$. By counting we can also see that $\dim_R \Lambda^k V \leq \binom{n}{k}$. We would like to show that $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}_{i_1 < \cdots < i_k}$ is a basis of $\Lambda^k V$ for $k \geq 1$. In particular $\Lambda^n V = \mathbb{R}\, v_1 \wedge \cdots \wedge v_n$.

In preparation for the proof that $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}_{i_1 < \cdots < i_k}$ is a basis note that we have k-linear and alternating maps

$$\varphi^{(k)} : V \times \cdots \times V \to \Lambda^*V, \quad \varphi^{(k)}(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k.$$

Proposition 21.6 (6.25 and 6.25.1 in online notes)**.** *For any vector spaces V, U and for any alternating k-linear map $f : V \times \cdots \times V \to U$ there exists a unique linear map $\bar{f} : \Lambda^k V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \ldots, v_n)$ (i.e. $f = \bar{f} \circ \varphi^{(k)}$). Hence $\operatorname{Hom}(\Lambda^k V, U) \xrightarrow{\simeq} \operatorname{Alt}^k(V, \ldots, V; U)$, $A \mapsto A \circ \varphi^{(k)}$ is an isomorphism.*

Proof. Since f is k-linear there exists a unique map $\tilde{f} : V^{\otimes k} \to U$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k)$. Since f is alternating $\tilde{f}|_{(I \cap V^{\otimes k})} = 0$. Hence there exists a unique map $\bar{f} : V^{\otimes k}/(I \cap V^{\otimes k}) = \Lambda^k V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_k) = \tilde{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k)$. □

Lemma 21.7 (6.26 in online notes)**.** *$\Lambda^{\dim V} V \neq 0$ hence $\Lambda^{\dim V} V = 1$.*

Proof. Let $n = \dim V$. We may assume that $V = \mathbb{R}^n$. Then for $\det : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we have

$$\det \left(\begin{array}{c|c|c|c|c} 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \end{array} \right) = 1$$

and so $\operatorname{Alt}^n(\mathbb{R}^n, \ldots, \mathbb{R}^n; \mathbb{R}) \neq 0$. Hence $\operatorname{Hom}(\Lambda^n V, \mathbb{R}) \neq 0$ and therefore $\Lambda^n V \neq 0$. □

Corollary 21.8 (6.26.1 in online notes)**.** *If $\{v_1, \ldots, v_n\}$ is a basis of V, then for any k with $1 \leq k \leq n$ the set $\{v_{i_1}, \ldots, v_{i_k}\}_{i_1 < \cdots < i_k}$ is a basis of $\Lambda^k V$.*

Proof. Suppose that $\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$ for some $\{a_{i_1, \ldots, i_k}\}_{i_1 < \cdots < i_k} \subseteq \mathbb{R}$. Fix $i_1^0 < \cdots < i_k^0$. Let $j_{k+1}^0 < \cdots < j_n^0$ be the complementary set of indices. Then

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{j_{k+1}}^0 \wedge \cdots \wedge v_{j_n}^0 = \begin{cases} \pm v_1 \wedge \cdots \wedge v_n & \text{if } (i_1, \ldots, i_k) = (i_1^0, \ldots, i_k^0) \\ 0 & \text{else} \end{cases}$$

Therefore

$$0 = \left(\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \right) \wedge v_{j_{k+1}}^0 \wedge \cdots \wedge v_{j_n}^0 = \pm a_{i_1, \ldots, i_k}$$

□

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