Last Time. For a finite dimensional vector space V we constructed an associative graded algebra:

$$\Lambda^* V = \bigoplus_{i=0}^{\dim V} \Lambda^i V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^{\dim V} V$$

Moreover, if $\{v_1, \ldots, v_n\}$ is a basis of V, then $\{v_{i_1} \wedge \cdots \wedge v_{i_l}\}_{i_1 < \cdots < i_k}$ is a basis of $\Lambda^k V$. Hence $\dim \Lambda^k V = \binom{n}{k}$.

Example 22.1. Let $V = (\mathbb{R}^3)^*$. Then $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ form a basis. In particular:

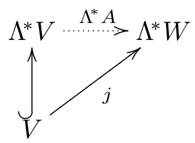
$$\begin{aligned} \mathrm{d}x(\vec{v}) &= v_1 \\ \mathrm{d}y(\vec{v}) &= v_2 \\ \mathrm{d}z(\vec{v}) &= v_3 \end{aligned}$$

Furthermore, $\Lambda^1(\mathbb{R}^3)^*$ is spanned by $\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x\}$ and $\Lambda^2(\mathbb{R}^3)^*$ is spanned by $\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$.

Proposition 22.2. *For any linear map $A : V \to W$ there exits a unique map of graded algebras $\Lambda^* A : \Lambda^* V \to \Lambda^* W$ such that for all $v_1, \ldots, v_k \in V$ we have:*

$$(\Lambda^* A)(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$

Proof. Consider the composite map $j : V \xrightarrow{A} W \hookrightarrow \Lambda^* W$. Then for all $v \in V$ we have $j(v) \cdot j(v) = 0$. So by the universal property of $V \hookrightarrow \Lambda^k V$ there exists a unique map of algebras $\Lambda^* A : \Lambda^* V \to \Lambda^* W$ such that the following diagram commutes:



□

Notation. Since $\Lambda^* A(\Lambda^k V) \subset \Lambda^k W$ it makes sense to set

$$\Lambda^k A := \Lambda^* A\big|_{\Lambda^k V}.$$

Our goal is to interpret $\Lambda^k(V^*)$ as alternating linear maps. More precisely we want to prove:

Lemma 22.3. *Let V be a finite dimensional vector space. For each $k \in \mathbb{Z}$ such that $1 \leq k \leq \dim V = n$ we have a natural isomorphism $\Lambda^k(V^*) \to \mathrm{Alt}^k(V, \ldots, V; \mathbb{R})$ such that $l_1 \wedge \cdots \wedge l_k \to [(v_1, \ldots, v_k) \mapsto \det(l_i(v_j))]$.*

Example 22.4. Consider $\mathrm{d}x \wedge \mathrm{d}y \in \Lambda^2(\mathbb{R}^3)^*$. By Lemma 22.3above

$$(\mathrm{d}x \wedge \mathrm{d}y)(\vec{v}, \vec{w}) = \det \begin{pmatrix} \mathrm{d}x(\vec{v}) & \mathrm{d}x(\vec{w}) \\ \mathrm{d}y(\vec{v}) & \mathrm{d}y(\vec{w}) \end{pmatrix} = v_1 w_2 - v_2 w_1$$

Example 22.5. Consider $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \in \Lambda^3(\mathbb{R}^3)^*$. Then

$$(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)(\vec{u}, \vec{v}, \vec{w}) = \det \begin{pmatrix} \mathrm{d}x(\vec{u}) & \mathrm{d}x(\vec{v}) & \mathrm{d}x(\vec{w}) \\ \mathrm{d}y(\vec{u}) & \mathrm{d}y(\vec{v}) & \mathrm{d}y(\vec{w}) \\ \mathrm{d}z(\vec{u}) & \mathrm{d}z(\vec{v}) & \mathrm{d}z(\vec{w}) \end{pmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w}),$$

the scalar triple product of the vectors $\vec{u}, \vec{v}, \vec{w}$.

Our proof of Lemma 22.3 uses *pairings.*

Definition 22.6. A *paring* of two vector spaces is a bilinear map $b : V \times W \to \mathbb{R}$. The pairing is *non-degenerate* if for all $v \in V$ and all $w \in W$ the following holds:

$$\Big[b(v_0, w) = 0 \quad \forall w \in W \Big] \Rightarrow v_0 = 0$$

$$\Big[b(v, w_0) = 0 \quad \forall v \in V \Big] \Rightarrow w_0 = 0$$

Remark 22.7. We've seen that $(\Lambda^k V)^* = \mathrm{Hom}(\Lambda^k V, \mathbb{R}) \simeq \mathrm{Alt}^k(V, \dots, V; \mathbb{R})$. So to prove Lemma 22.3 it's enough to produce an isomorphism $(\Lambda^k V)^* \simeq \Lambda^k(V^*)$. We'll do so by constructing a pairing $\Lambda^k(V*) \times \Lambda^k V \to \mathbb{R}$ such that $b(l_1 \wedge \cdots \wedge l_n, v_1 \wedge \cdots \wedge v_n) = \det(l_i(v_j))$.

Example 22.8 (of pairings)**.** $V^* \times V \to \mathbb{R}$ such that $(l, v) \mapsto l(v)$ is a non-degenerate pairing.

Lemma 22.9. *If $b : V \times W \to \mathbb{R}$ is a non-degenerate pairing, then the linear maps*

$$\begin{aligned} b_1^\# &: V \to W^*, & b_1^\#(v)(-) &\stackrel{def}{=} b(v, -) \\ b_2^\# &: W \to V^*, & b_2^\#(w)(-) &\stackrel{def}{=} b(-, w) \end{aligned}$$

are isomorphisms.

Proof. By dimension count it is enough to check injectivity. By non-degeneracy of the pairing b we have:

$$b_1^\#(v) = 0 \iff b(v, w) = 0 \Rightarrow v = 0$$

Therefore $b_1^\#$ is injective. Similarly, $b_2^\#$ is injective. □

Proof of Lemma 22.3. Consider the following $2k$-linear map

$$\begin{aligned} \overbrace{V^* \times \cdots \times V^*}^{k} \times \overbrace{V \times \cdots \times V}^{k} &\to \mathbb{R} \\ (l_1 \wedge \cdots \wedge l_k, v_1 \wedge \cdots \wedge v_l) &\mapsto \det(l_i(v_j)) \end{aligned}$$

Then for each fixed k-tuple $(v_1, \dots, v_k)$ we have an alternating k-linear map $\Lambda^k V^* \times \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ such that $(l_1 \wedge \cdots \wedge l_k, v_1 \wedge \cdots \wedge v_l) \mapsto \det(l_i(v_j))$. Freezing the left hand side gives a map $\Lambda^k V^* \times \Lambda^k V \to \mathbb{R}$ such that $(l_1 \wedge \cdots \wedge l_k, v_1 \wedge \cdots \wedge v_l) \mapsto \det(l_i(v_j))$. We get a pairing. Now let us prove non-degeneracy. $b : V \times W \to \mathbb{R}$ is non-degenerate if there exists a basis $\{v_1, \dots, v_n\}$ of V and a basis $\{w_1, \dots, w_n\}$ of W such that $b(v_i, w_j) = \delta_{ij}$. In our case pick a basis $\{e_1, \dots, e_n\}$ of V^*. Let $\{e_1^*, \dots, e_n^*\}$ denote the dual basis. Then $\{e_{i_1}, \dots, e_{i_k}\}$ is a basis of $\Lambda^k V$ and $\{e_{i_1}^*, \dots, e_{i_k}^*\}$ is a basis of $\Lambda^k V^*$. Note that

$$e_{j_1}^*(e_{i_1}) \cdots e_{j_k}^*(e_{i_k}) = \begin{cases} 1 & \text{if } (i_1, \dots, i_k) = (j_1, \dots, j_k) \\ 0 & \text{else} \end{cases}$$

Therefore

$$b(e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*, e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{\sigma \in S_k} (-1)^r e_{j_1}^*(e_{i_{\sigma(1)}}) = 1 + 0 + 0 + \cdots + 0 = 1.$$

□

Typeset by R. S. Kueffner II