Last Time. We constructed for every finite dimensional vector space V a non-degenerate bilinear pairing

$$\Lambda^k(V*) \times \Lambda^k V \to \mathbb{R} \quad \text{with} \langle l_1 \wedge \cdots \wedge l_k, v_1 \wedge \cdots \wedge v_k\rangle = \det(l_i(v_j))$$

for any $l_1, \ldots, l_k \in V^*$, any $v_1, \ldots, v_k \in V$. As a consequence we can identify any product $l_1 \wedge \cdots \wedge l_k \in \Lambda^k(V*)$ with an alternating k-linear map that whose value on the k-tuple $v_1, \ldots, v_k$ is $\det(l_i(v_j))$. From now one we identify

$$\Lambda^k(V^*) \xrightarrow{\sim} \mathrm{Alt}^k(V; \mathbb{R}),$$

where $\mathrm{Alt}^k(V;\mathbb{R})$ denotes the space of alternating k-linear maps. As a result, since $\Lambda^*(V^*)$ is a graded algebra, we can now now multiply k-linear and ℓ-linear alternating maps and get $k+\ell$-linear alternating maps:

$$\begin{aligned} \Lambda^k(V^*) \times \Lambda^l(V^*) &\longrightarrow \Lambda^{k+l}(V^*) \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta. \end{aligned}$$

Recall. For a manifold M, the *tangent bundle* $TM = \coprod_{q\in M} T_q M$ can be given the structure of a manifold. In particular if $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a chart on M, then

$$\tilde{\varphi} = (x_1, \ldots, x_m, \mathrm{d}x_1, \ldots, \mathrm{d}x_m) : TU \to \mathbb{R}^m \times \mathbb{R}^m$$

is a chart on TM. Recall also that Vector fields on M are *sections* of $TM \xrightarrow{\pi} M$, that is, maps $X : M \to TM$ with $\pi \circ X = id_M$.

Similarly one can define the *cotangent bundle* $T^*M = \coprod_{q\in M} T_q^* M$ and give it the structure of a smooth manifold in more or less the same way we made the tangent bundle into a manifold. That is, we manufacture new coordinate charts on T^*M out of coordinate charts on M and check that the transition maps between the new coordinate charts are smooth. If φ is a chart then

$$\overline{\varphi} = \left(x_1, \ldots, x_m, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right) : T^*U \to \mathbb{R}^m \times \mathbb{R}^m$$

$$(q, \eta) \mapsto \left(x_1(q), \ldots, x_m(q), \left\langle \eta, \left.\frac{\partial}{\partial x_1}\right|_q \right\rangle, \ldots, \left\langle \eta, \left.\frac{\partial}{\partial x_m}\right|_q \right\rangle\right)$$

is a chart on T^*M.

Here is an excerpt from the old notes checking the smoothness of the transition maps. Let $\psi = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ be a coordinate chart on M with $V \cap U \neq \emptyset$. Then

$$\begin{aligned} \bar{\psi} \circ \bar{\phi}^{-1}(r_1, \ldots, r_n, w_1, \ldots, w_n) =& \bar{\psi}(\sum_{i=1}^n w_i (dx_i)_{\phi^{-1}(r)}) \\ =& ((\psi \circ \phi^{-1})(r), \frac{\partial}{\partial y_1}(\sum_{i=1}^n w_i\, dx_i), \ldots, \frac{\partial}{\partial y_n}(\sum_{i=1}^n w_i\, dx_i)) \\ =& ((\psi \circ \phi^{-1})(r), \sum_i w_i \frac{\partial x_i}{\partial y_1}, \ldots, \sum_i w_i \frac{\partial x_i}{\partial y_n}). \end{aligned}$$

We conclude that

$$\bar{\psi} \circ \bar{\phi}^{-1}(r_1, \cdots, r_n, w_1, \cdots, w_n) = (\psi \circ \phi^{-1}(r), \left(\frac{\partial x_i}{\partial y_j}(r)\right) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}), \tag{23.1}$$

which is smooth. The rest of the argument proceeds as in the case of the tangent bundle.

We will see in a week that one can construct the exterior powers $\Lambda^k(T^*M) = \coprod_{q\in M} \Lambda^k(T_q^*M)$ and get manifolds over M. The smooth sections of $\Lambda^k(T^*M) \to M$ are called differential k-forms.

Notation. $\Omega^k(M) \stackrel{\text{def}}{=} \Gamma(\Lambda^k(T^*M))$ such that $\pi \circ \omega = \mathrm{id}_M$. What then is $\Omega^0(M)$? By convention $\Lambda^0(T^*M) = M \times \mathbb{R}$. Consequently

$$\Omega^0(M) = \{M \xrightarrow{\tau} M \times \mathbb{R} \mid \tau(q) = (q, f(q)),\ f : M \to \mathbb{R}\} = C^\infty(M)$$

Remark 23.1. Differential forms on M can be multiplied point-wise: $\forall \alpha \in \Omega^k(M)\ \forall \beta \in \Omega^l(M)$

$$(\alpha \wedge \beta)_q \stackrel{\text{def}}{=} \alpha_q \wedge \beta_q$$

for all points $q \in M$.

Example 23.2. Let $M = \mathbb{R}^m$. Then $TM = \mathbb{R}^m \times \mathbb{R}^m$ and $T^*M = \mathbb{R}^m \times (\mathbb{R}^m)^*$. At every $q \in \mathbb{R}^m$ we have a basis of $T_q^*\mathbb{R}^n : (\mathrm{d}x_1)_q, \ldots, (\mathrm{d}x_m)_q$. So $\Lambda^k(T^*M) = \mathbb{R}^m \times \Lambda^k((\mathbb{R}^m)^*)$ and

$$\alpha \in \Omega^k(\mathbb{R}^m) \Leftrightarrow \alpha = \sum_{|I|=k} a_I \,\mathrm{d}x_I$$

where $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, m\}$, $a_I \in C^\infty(\mathbb{R}^M)$, and $\mathrm{d}x_I \stackrel{\text{def}}{=} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m}$

Example 23.3. 1-forms on $\mathbb{R}^2$ look like

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$$

where $M(x,y)$, $N(x,y)$ are smooth functions.

Example 23.4. 2-forms on $\mathbb{R}^3$ look like

$$P(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y + Q(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + R(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x,$$

where, again, P, Q and R are smooth functions.

Example 23.5.

$$\begin{aligned}
\alpha &= \cos v\,\mathrm{d}u - u \sin v\,\mathrm{d}v \in \Omega^1(\mathbb{R}^2) \\
\beta &= \sin v\,\mathrm{d}u + u \cos v\,\mathrm{d}v \in \Omega^1(\mathbb{R}^2)
\end{aligned}$$

$$\begin{aligned}
\alpha \wedge \beta &= (\cos v\,\mathrm{d}u - u \sin v\,\mathrm{d}v) \wedge (\sin v\,\mathrm{d}u + u \cos v\,\mathrm{d}v) \\
&= \cos v \sin v\,\mathrm{d}u \wedge \mathrm{d}u + u \cos^2 v\,\mathrm{d}u \wedge \mathrm{d}v \\
&- u \sin^2 v\,\mathrm{d}v \wedge \mathrm{d}u + u^2 \sin v \cos v\,\mathrm{d}v \wedge \mathrm{d}v \\
&= u \cos^2 v\,\mathrm{d}u \wedge \mathrm{d}v - u \sin^2 v\,\mathrm{d}v \wedge \mathrm{d}u \\
&= u \cos^2 v\,\mathrm{d}u \wedge \mathrm{d}v + u \sin^2 v\,\mathrm{d}u \wedge \mathrm{d}v \\
&= u\,\mathrm{d}u \wedge \mathrm{d}v
\end{aligned}$$

Remark 23.6. For any function $f \in C^\infty(M)$, $\mathrm{d}f$ is a 1-form. In particular given a coordinate chart $(x_1, \ldots, x_m)$ we have

$$\mathrm{d}f = \sum \left\langle \mathrm{d}f, \frac{\partial}{\partial x_i} \right\rangle \mathrm{d}x_i = \sum \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$$

Example 23.7.

$$\begin{aligned}
f(u,v) &= u \cos v \qquad & \mathrm{d}f &= \mathrm{d}(u \cos v) = \cos v\,\mathrm{d}u - u \sin v\,\mathrm{d}v \\
g(u,v) &= u \sin v \qquad & \mathrm{d}g &= \mathrm{d}(u \sin v) = \cos v\,\mathrm{d}u + u \sin v\,\mathrm{d}v
\end{aligned}$$

Hence in Example 23.5 we have $\mathrm{d}f \wedge \mathrm{d}g = \mathrm{d}(u \cos v) \wedge \mathrm{d}(u \sin v) = u\,\mathrm{d}u \wedge \mathrm{d}v$.

Remark 23.8. Once we define *pullback* of differential forms we'll see that $u\,\mathrm{d}u \wedge \mathrm{d}v$ is the pullback of $\mathrm{d}x \wedge \mathrm{d}y$ by $f(u,v) = (u \cos v, u \sin v)$.

We now proceed to define pullbacks of differential forms by smooth maps.

Recall. Given a linear map $A : V \to W$ between two vector spaces we get $\Lambda^k A : \Lambda^k V \to \Lambda^k W$. We also have $A^* : W^* \to v^*$ where

$$(A * l)(v) = l(Av) = (l \circ A)(v)$$

Hence given a linear map $A : V \to W$ we get $\Lambda^k(A^*) : \Lambda^k W^* \to \Lambda^k V^*$ with $(l_1 \wedge \cdots \wedge l_k) \mapsto (A^* l_1) \wedge \cdots \wedge (A^* l_k)$.

What does this map $\Lambda^k(A^*)$ amount to when we identify exterior powers of the dual vector spaces with alternating multilinear maps? We compute:

$$\begin{aligned}((\Lambda^k(A^*))l_1 \wedge \cdots \wedge l_k)(v_1, \ldots, v_k) &= (l_1 \circ A) \wedge \cdots \wedge (l_k \circ A)(v_1, \ldots, v_k) \\ &= \det(l_i(Av_j)) \\ &= (l_1 \wedge \cdots \wedge l_k)(Av_1, \ldots, Av_k)\end{aligned}$$

Remark 23.9. Note that for all $\alpha \in \Lambda^k(W^*)$ and all $\beta \in \Lambda^n(W^*)$ we have

$$\Lambda^k(A^*)\alpha \wedge \Lambda^n(A^*)\beta = \Lambda^{k+n}(A^*)(\alpha \wedge \beta)$$

since $\Lambda^*(A^*)$ is a map of algebras!

With these preliminaries out of the way we are now set to define pullbacks of differential forms. If $F : M \to N$ is a map of manifolds then for all $q \in M$ we have $\mathrm{d}F_q : T_qM \to T_{F(q)}N$ which gives us the following map of algebras

$$\Lambda^*((\mathrm{d}F_q)^*) : \Lambda^*(T^*_{F(q)}N) \to \Lambda^*(T^*_qM)$$

So for $\alpha \in \Omega^k(N)$ we get $F^*\alpha \in \Omega^k(M)$ defined by

$$(F^*\alpha)_q = \Lambda^*((\mathrm{d}F_q)^*)\alpha_{F(q)}. \tag{23.2}$$

Strictly speaking we should check that if α is a *smooth* differential form on N then its pullback $F^*\alpha$ is also smooth. But let's not worry about this for the time being, certainly not until after we define $\Lambda^k(T^*M)$. Equation (23.2) translates into:

$$(F^*\alpha)_q(v_1, \ldots, v_k) = \alpha_{F(q)}((\mathrm{d}F_q)v_1, \ldots, (\mathrm{d}F)_q) \tag{23.3}$$

Why did we define pullback of differential forms by (23.2) and not by (23.3)? Because (23.2) automatically implies that

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$$

for all differential forms α, β on N.

Next Time. $F^*(\mathrm{d}f) = \mathrm{d}f \circ F$. Hence if $x = r\cos\theta$ and $y = r\sin\theta$ then $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$.

Typeset by R. S. Kueffner II. Edited by E.L.