Last Time.

(1) We "defined" differential forms

$$\Omega^*(M) \stackrel{\text{def}}{=} \Omega^0(M) \oplus \cdots \oplus \Omega^{\dim M}(M)$$

where $\Omega^0(M) = C^\infty(M)$ and

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M)) = \{w : M \to \Lambda^k(T^*M) \mid w_q \in \Lambda^k(T^*_qM)\ \forall q \in M\}$$

(2) Given a map between manifolds $F : M \to N$ we defined the pullback $F^* : \Omega^*(N) \to \Omega^*(M)$, a map of algebras as follows. This definition amounts to the following. If f is a zero form (i.e., a function) then

$$F^*f = f \circ F.$$

For all $k > 0$ and all $\alpha \in \Omega^k(N)$

$$(F^*\alpha)_q(v_1, \dots, v_k) = \alpha_{F(q)}(\mathrm{d}F_q v_1, \dots, \mathrm{d}F_q v_k),$$

for all $q \in M$, $v_1, \dots, v_k \in T_qM$. (This is not how one *computes* pullbacks.)

The goal of today's lecture is to define integration of compactly supported differential forms on oriented manifolds. We first observe:

Proposition 24.1. *If $U, V \in \mathbb{R}^n$ are open sets and $F : U \to V$ is a C^∞ map then $F^*(f(x)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = [f \circ F(x)]\det(\mathrm{d}F_x)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.*

Proof. Homework 8 problem 4. □

Definition 24.2. For a k-form $\mu \in \Omega^k(M)$ we define its support $\mathrm{supp}(\mu)$ to be the closure of the set of points where the form is nonzero:

$$\mathrm{supp}(\mu) \stackrel{\text{def}}{=} \overline{\{q \in M \mid \mu_q \neq 0\}}.$$

Notation. $\Omega^k_c = \{\mu \in \Omega^k(M) \mid \mathrm{supp}(\mu) \text{ is compact}\}$, the space of compactly supported k-forms.

Definition 24.3. A manifold M is *orientable* if there exists an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}$ such that for all α and β we have $\det\big(\mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1})\big) > 0$.

Definition 24.4. A choice of such an atlas in Definition 24.3 is an *orientation.*

Definition 24.5. Two orientations are the same if their union is also an orientation.

Example 24.6. $\mathbb{R}^n$ has a canonical orientation $\{\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n\}$.

Example 24.7. $\{\varphi : \mathbb{R}^n \to \mathbb{R}^n\}$ such that $\varphi(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ is an orientation of $\mathbb{R}^n$ that is not the canonical orientation.

Example 24.8. $\{\psi : \mathbb{R}^n \to \mathbb{R}^n \mid n > 1\}$ such that $\psi(x_1, \dots, x_n) = (x_2, x_1, x_3, \dots, x_n)$ is an orientation which is the same as the one in Example 24.7.

Given an oriented manifold M with $\dim M = m$ we will define a linear map $\int_M : \Omega^m_c(M) \to \mathbb{R}$, the integration map, in several steps.

Step 1. $U \subseteq \mathbb{R}^m$ open, $\mu \in \Omega^m_c(M)$, $\mu = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$, and $f \in C^\infty$, then

$$\int_U f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m \stackrel{\text{def}}{=} \int_U f(x)\,\mathrm{d}x_1 \cdots \mathrm{d}x_m$$

Step 2. M a manifold, $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}$ an atlas which orients M, $\mu \in \Omega^m_c(M)$ with $\mathrm{supp}\,\mu \subseteq U_\alpha$ for some α, then

$$\int_M \mu \stackrel{\text{def}}{=} \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*\mu$$

To make sure that this definition makes sense we need to prove

Proposition 24.9. *Suppose $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ and $\psi : U_\alpha \to \mathbb{R}^m$ are two coordinate chart with $\det \mathrm{d}(\psi \circ \varphi_\alpha^{-1}) > 0$. Then*

$$\int_{\psi(U_\alpha)} (\psi^{-1})^*\mu = \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*\mu$$

Proof. It is enough to show that for every diffeomorphism $F : W \to V$, $\forall W, V \subseteq \mathbb{R}^m$, $\forall \nu \in \Omega_c^m(V)$

$$\int_W F^*\nu = \int_{V=F(W)} \nu$$

Let $\nu = f(y)\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_m$ for some $f \in C_c^\infty(V)$. Then

$$\int_V \nu = \int_V f(y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_m$$

But $F^*\nu = f(F(x))\det(\mathrm{d}F_x)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$ and so

$$\begin{aligned}\int_V F^*\nu &= \int_W f(F(x))\det(\mathrm{d}F_x)\,\mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &= \int_W f(F(x))\big|\det(\mathrm{d}F_x)\big|\,\mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &= \int_W f(y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_m\end{aligned}$$

where the last equality is holds by the change of variables formula. □

Step 3. General case: Let $\mu \in \Omega_c^m(M)$ be arbitrary. Pick an atlas which gives M its orientation, say $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha\in A}$. Since $\operatorname{supp}\mu$ is compact $\exists \alpha_1, \ldots, \alpha_k$ such that $\operatorname{supp}\mu \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. Let $U_0 = M \setminus \operatorname{supp}\mu$. Let $\{\rho_0, \ldots, \rho_k\}$ be a partition of unity subordinate to $\{U_0, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$, i.e. $\operatorname{supp}\rho_0 \subseteq M \setminus \operatorname{supp}\mu$, $\operatorname{supp}\rho_j \subseteq U_{\alpha_j}$ for $j \geq 1$. Now define

$$\int_M \mu \stackrel{\text{def}}{=} \sum_{j=1}^k \int_{U_{\alpha_j}} \rho_j M$$

We need to check that the right-hand side does not depend on choices. Suppose that $\{\psi_i : V_i \to \mathbb{R}^m\}_{i=1}^l$ is another collection of coordinate charts with $\operatorname{supp}\mu \subseteq \bigcup_{i=1}^l V_i$ (with $V_0 = M \setminus \operatorname{supp}\mu$). $\{\tau_i\}_{i=0}^l$ is a partition of unity subordinate to $\{V_i\}_{i=0}^l$ and $\det\big(\mathrm{d}(\varphi\alpha_j \circ \psi_i^{-1})\big) > 0$ for all i, j. Then

$$\int_{\psi_i(V_i \cap U_{\alpha_j})} (\psi_i^{-1})^*\rho_j\tau_i\mu = \int_{\varphi_{\alpha_j}(V_i\cap U_{\alpha_j})} (\varphi_{\alpha_j}^{-1})^*\rho_j\tau_i\mu$$

by Proposition 24.9 and therefore

$$\begin{aligned}\sum_j \int_{U_{\alpha_j}} (\rho_j\mu) &= \sum_j \int_{\varphi_{\alpha_j}(U_{\alpha_j})} (\varphi_{\alpha_j})^*(\rho_j\mu) \\ &= \sum_j \int_{\varphi_{\alpha_j}(U_{\alpha_j})} (\varphi_{\alpha_j})^*\rho_j(\sum_i \tau_i\mu) \\ &= \sum_{i,j} \int_{\varphi_{\alpha_j}(U_{\alpha_j}\cap V_i)} (\varphi_{\alpha_j})^*\rho_j\tau_i\mu \\ &= \sum_{i,j} \int_{\psi_i(U_{\alpha_j}\cap V_i)} (\psi_i^{-1})^*\rho_j\tau_i\mu \\ &= \sum \int_{V_i} \tau_i\mu\end{aligned}$$

We conclude that $\int_M : \Omega_c^m(M) \to \mathbb{R}$ is well-defined. Note that it does depend on the choice of an orientation.

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