Last Time. We defined orientations and a linear map $\int_M : \Omega_c^{\dim M} \to \mathbb{R}$ for oriented manifolds M.

How do we actually compute $\int_M \mu$?

Lemma 25.1. *$\forall f \in C^\infty(N)$ $\forall F : M \to N$ we have $F^* \mathrm{d}f = \mathrm{d}(F^* f)$*

Proof. Fix $q \in M$, $v \in T_q M$. Recall that $\forall h \in C^\infty(M)$ we have $(\mathrm{d}h_q)(v) = v(h)$. Therefore

$$\mathrm{d}(F^* f)_q(v) = v(f \circ F) = (\mathrm{d}F_q v)(f) = \mathrm{d}f_{F(q)}(\mathrm{d}F_q v) = (F^* \mathrm{d}f)_q(v)$$

□

Fact 25.2. *If $N \subseteq M$ is a codimension 1 submanifold, then $\forall \mu \in \Omega_c^{\dim M}(M)$ we have $\int_M \mu = \int_{M \setminus N} \mu$.*

Example 25.3. Compute

$$\int_{\mathbb{S}^1} -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

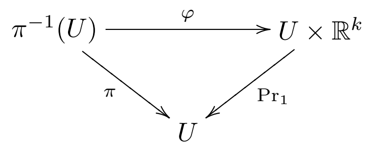
for $\mathbb{S}^1 \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ oriented by the parametrization $f : \theta \mapsto (\cos\theta, \sin\theta)$ for $0 < \theta < 2\pi$. Then

$$\begin{aligned}
\int_{\mathbb{S}^1} -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y &= \int_{(0,2\pi)} f^*\left(-\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y\right) \\
&= \int_{(0,2\pi)} -\frac{\sin\theta}{\cos^2\theta + \sin^2\theta}\,\mathrm{d}(\cos\theta) + \frac{\cos\theta}{\cos^2\theta + \sin^2\theta}\,\mathrm{d}(\sin\theta) \\
&= \int_{(0,2\pi)} \sin^2\theta\,\mathrm{d}\theta + \cos^2\theta\,\mathrm{d}\theta \\
&= \int_{(0,2\pi)} \mathrm{d}\theta = 2\pi
\end{aligned}$$

We still need to define differential forms. As a first step we define vector bundles.

Definition 25.4. A (real) *vector bundle* E of rank k over a manifold M is a manifold E together with a smooth map $\pi = \pi_E : E \to M$ such that the following two conditions hold.

(1) For each $q \in M$ the fiber $E_q \stackrel{\text{def}}{=} \pi^{-1}(q)$ is a real vector space of dimension k.
(2) For each $q \in M$ there exists an open neighborhood U of q and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ so that
  (a) $\forall p \in U$, $\varphi(E_p) = \{p\} \times \mathbb{R}^k$. I.e. the following diagram commutes:



  (b) $\forall p \in U$, $\varphi\big|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ are $\mathbb{R}$-linear isomorphisms.

Terminology.

- E is called the *total space* of the vector bundle.
- M is called the *base.*
- $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is called a *local trivialization.*
- $E_p = \pi^{-1}(p)$ is called the *fiber* above $p \in M$.
- $\mathbb{R}^k$ is called the *typical fiber.*

Remark 25.5. Replacing "real" by "complex" in the definition above gives a definition of a complex vector bundle.

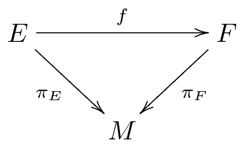
Remark 25.6. Instead of $\mathbb{R}^k$ in the definition of the vector bundle we could have a fixed finite dimensional vector space V (i.e., we don't really need to choose a basis).

Example 25.7. $M \times \mathbb{R}^k \to M$, $(p, v) \mapsto p$ is a vector bundle of rank k. $\varphi = \mathrm{id}$ and $U = M$.

Example 25.8. M any manifold. $TM \xrightarrow{\pi} M$ is a vector bundle of rank equal to $\dim M$. Local trivializations? If $\varphi = (x_1, \dots, x_n) : U \to \mathbb{R}^m$ is a coordinate chart on M, then $\pi^{-1}(U) = TU \xrightarrow{\varphi} U \times \mathbb{R}^m$, $(p, v) \mapsto (p, (\mathrm{d}x_1)_p v, \dots, (\mathrm{d}x_m)_p)$ is a local trivialization.

Definition 25.9. Let $\pi_E : E \to M$, $\pi_F : F \to M$ be two vector bundles over M. A *map of vector bundles* $f : E \to F$ is a map of manifolds $f : E \to F$ so that

(1) $\forall q \in M$, $f(E_q) \subseteq F_q$. I.e. the following diagram commutes:



(2) $\forall q \in M$, $f\big|_{E_q} : E_q \to F_q$ is linear.

Definition 25.10. A map of vector bundles (over M) $f : E \to F$ is an *isomorphism* if there exists a map of vector bundles $g : F \to E$ such that $g \circ f = \mathrm{id}_E$ and $f \circ g = \mathrm{id}_F$.

Exercise 25.1. Any bijective map of vector bundles is an isomorphism. *Hint:* Do it for trivial bundles first.

Definition 25.11. A vector bundle $\pi : E \to M$ is *trivial* if it is isomorphic to a bundle $M \times V \to M$ for some vector space V.

Example 25.12. For any Lie group $\mathfrak{g}$, $T\mathfrak{g} \to \mathfrak{g}$ is trivial. The map $\mathfrak{g} \times T_{\mathrm{id}}\mathfrak{g} \to T\mathfrak{g}$, $(g, v) \mapsto (g, (\mathrm{d}L_g)_{\mathrm{id}} v)$ is an isomorphism. The inverse is given by $h(g, w) = (g, (\mathrm{d}L_g)_g w)$.

Fact 25.13. *$T\mathbb{S}^2 \to \mathbb{S}^2$ is not trivial.*

Remark 25.14. If $E \xrightarrow{\pi} M$ is a vector bundle and $W \subseteq M$ is open, then $E\big|_W = \pi^{-1}(W) \to W$ is also a vector bundle. Note that if $E \to M$ is a vector bundle, then for all $p \in M$ there exists a neighborhood U of p such that $E\big|_U$ is trivial. "Vector bundles are locally trivial."

Definition 25.15. A *section* s of a vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

Example 25.16. A section of $TM \to M$ is a vector field.

Example 25.17. A section of $T^*M \to M$ is a 1-form.

Example 25.18. A section of $M \times V \to M$, where V is a vector space, is a V-valued function on M.

Remark 25.19. The space of section $\Gamma(E)$ of a vector bundle $E \to M$ is a $C^\infty(M)$-module: we can add two smooth section and get a smooth section; we can multiply a smooth section by a smooth function and get a smooth section.

Definition 25.20. A *local section* of a vector bundle is a section of $E\big|_W \to W$ for some open $W \subseteq M$.

Example 25.21. Consider $\mathbb{S}^2 \overset{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$. The normal bundle of the embedding $\mathbb{S}^2 \subset \mathbb{R}^3$ is

$$E = \{(q, v) \in \mathbb{S}^2 \times \mathbb{R}^3 \mid v = \lambda q \text{ for some } \lambda \in \mathbb{R}\}$$

Typeset by R. S. Kueffner II