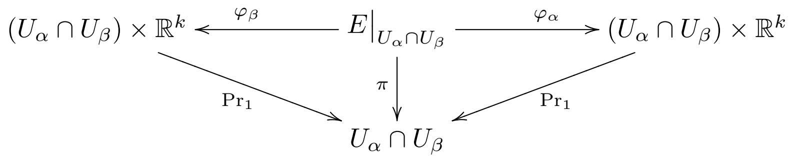
Last Time.

(1) Defined vector bundles $E \xrightarrow{\pi} M$.
(2) Space of sections $\Gamma(E)$ of $E \xrightarrow{\pi} M$.
(3) Local sections.
(4) Morphisms of vector bundles.
(5) Trivial bundles.
(6) Vector bundles are locally trivial.

Goal. Operations on vector bundles. I.e. given $E \xrightarrow{\pi} M$ we will construct $E^* \xrightarrow{\pi^*} M$ with $(E^*)_q = \mathrm{Hom}(E_q, \mathbb{R})$ or $\Lambda^k(E) \to M$, et cetera.

Let $E \xrightarrow{\pi} M$ be a rank k vector bundle. Then there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M and local trivializations $\{\varphi_\alpha : E\big|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k\}_{\alpha\in A}$ with $\mathrm{pr}_1 \circ \varphi_\alpha = \pi$ and $\forall q \in U_\alpha$, $\varphi_\alpha\big|_{E_q} : E_q \to \{q\}\times\mathbb{R}^k$ are linear isomorphisms. What happens over the overlaps $U_\alpha \cap U_\beta$? The following diagram commutes:



Hence $\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta)\times\mathbb{R}^k \to (U_\alpha\cap U_\beta)\times\mathbb{R}^k$ is $C^\infty$and is of the form $(q,v)\mapsto(q,F(q,v))$ for some smooth map $F : (U_\alpha \cap U_\beta)\times\mathbb{R}^k \to \mathbb{R}^k$ with $F(q,-) : \mathbb{R}^k \to \mathbb{R}^k$ a linear isomorphism for all $q \in (U_\alpha \cap U_\beta)$. So there is a smooth map $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ with $F(q,v) = \varphi_{\alpha\beta}(q)v$. That is, $(\varphi_\alpha \circ \varphi_\beta^{-1})(q,v) = (q, \varphi_{\alpha\beta}(q)v)$. Thus we get a collection of maps $\{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)\}_{\alpha,\beta\in A}$ called the *transition maps* of the bundle $E \xrightarrow{\pi} M$ with respect to the cover $\{U_\alpha\}_{\alpha\in A}$.

Example 26.1. $E = TM \xrightarrow{\pi} M$, $\{\phi^{(\alpha)} = (x_1^{(\alpha)}, \ldots, x_m^{(\alpha)}) : U_\alpha \to \mathbb{R}^m\}$ an atlas of coordinate charts. Corresponding trivializations $\varphi_\alpha : TU_\alpha \to U_\alpha \times \mathbb{R}^m$ are given by $\varphi_\alpha(q,v) = (q, (\mathrm{d}x_1^{(\alpha)})_q v, \ldots, (\mathrm{d}x_m^{(\alpha)})_q v)$.

$$\begin{aligned}
\varphi_{\alpha\beta}(q)w &= (\mathrm{pr}_2 \circ \varphi_\alpha \circ \varphi_\beta^{-1})(q,v) \\
&= (\mathrm{pr}_2 \circ \varphi_\alpha)\left(q, \sum w_j \frac{\partial}{\partial x_j^{(\beta)}}\bigg|_q\right) \\
&= \left((\mathrm{d}x_1^{(\alpha)})q \sum w_j \frac{\partial}{\partial x_j^{(\beta)}}\bigg|_q, \ \ldots,\ (\mathrm{d}x_m^{(\alpha)})q \sum w_j \frac{\partial}{\partial x_j^{(\beta)}}\bigg|_q, \right) \\
&= \left(\frac{\partial x_i^{(\alpha)}}{\partial x_j^{(\beta)}}(q)\right)\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}
\end{aligned}$$

Therefore the transition maps are $\varphi_{\alpha\beta}(q) = \mathrm{d}(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(q)}$

Example 26.2. Tautological line bundle $L \to \mathbb{RP}^n$.
Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{o\})/\sim$ is the space of all lines (through 0) in $\mathbb{R}^{n+1}$ where $v \sim \tilde{v}$ if and only if v and $\tilde{v}$ are linearly independent. Hence $L = \{(l,w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \bigm| w \in l\}$. We have a map $\pi : L \to \mathbb{RP}^n$, $\pi(l,w) = l$. We won't check that this is a vector bundle.

Let $U_i = \{[v_0, \ldots, v_n] \in \mathbb{RP}^n \bigm| v_i \neq 0\}$. Then $\varphi_i : L_{U_i} \to U_i \times \mathbb{R}$ such that

$$\varphi_i([v_0,\ldots,v_n],(w_0,\ldots,w_n)) - ([v_0,\ldots,v_n],w_i)$$

are local trivializations. It is not hard to see that

$$\varphi_j^{-1}([v_0,\ldots,v_n],\lambda)\left([v_0,\ldots,v_n], \lambda\cdot\left(\frac{v_0}{v_j},\ldots,\frac{v_n}{v_j}\right)\right).$$

Consequently

$$(\varphi_i \circ \varphi_j^{-1})([v], \lambda) = \left([v], \lambda \frac{v_i}{v_j}\right)$$

Therefore $\varphi_{ij}([v]) = \frac{v_i}{v_j}$ are the transition maps.

Remark 26.3. Note that the transition maps $\{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ satisfy a number of conditions. Since $\varphi_{\alpha\beta}(q)v = (\varphi_\alpha \circ \varphi_\beta^{-1})(q, v)$ we have the following three conditions:

- $\varphi_{\alpha\alpha}(q) = \mathrm{id}_{\mathbb{R}^k}$
- $\varphi_{\alpha\beta} \cdot \varphi_{\beta\alpha} = \mathrm{id}_{\mathbb{R}^k}$
- $\varphi_{\alpha\beta} \cdot \varphi_{\beta\gamma} \cdot \varphi_{\gamma\alpha} = \mathrm{id}_{\mathbb{R}^k}$

These conditions are called the cocycle conditions.

Goal. Let us now construct the bundle $E^* \to M$ dual to a bundle $E \to M$.

As a set we define $E^* \overset{\text{def}}{=} \coprod_{q \in M} (E_q)^*$. We need to give E^* a manifold structure and we need local trivializations. Since $E \xrightarrow{\pi} M$ is a vector bundle, we have a cover $\{U_\alpha\}$ of M and local trivializations $\varphi : E\big|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ where $k = \operatorname{rank} E$. For all $q \in U_\alpha$ we have $\varphi_\alpha\big|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$. Hence we have

$$(\varphi_\alpha\big|_{E_q})^* : \{q\} \times (\mathbb{R}^k)^* \to (E_q)^* = E_q^*$$

and therefore $\forall q \in U_\alpha$

$$\left[(\varphi_\alpha\big|_{E_q})^*\right]^{-1} : E_q^* \to \{q\} \times (\mathbb{R}^k)^*$$

We then get a collection of bijections $\{\varphi_\alpha^* : E^*\big|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^k)^*\}_{\alpha \in A}$. What about transition maps? For $w \in \mathbb{R}^k$ and $l \in (\mathbb{R}^k)^*$ we have

$$\begin{aligned}(\varphi_\alpha \circ \varphi_\beta)^{-1}(q, w) &= (q, \varphi_{\alpha\beta}(q)w) \\ &= \left(q, \varphi_\alpha\big|_{E_q} \circ \left(\varphi_\beta\big|_{E_q}\right)^{-1} w\right).\end{aligned}$$

Consequently

$$(\varphi_\alpha^* \circ (\varphi^*)_\beta)^{-1}(q, l) = \left(q, \left[(\varphi_\alpha\big|_{E_q})^*\right]^{-1} \left(\left[(\varphi_\beta\big|_{E_q})^*\right]^{-1}\right)^{-1} l\right).$$

Hence the purported transition maps $\varphi_{\alpha\beta}^*$ for E^* are given by $\varphi_{\alpha\beta}^*(q) = [\varphi_{\alpha\beta}(q)^*]^{-1}$. Therefore $\varphi_{\alpha\beta}^* = G \circ \varphi_{\alpha\beta}$ where G is the composite map

$$\begin{array}{ccccc} \mathrm{GL}(\mathbb{R}^k) & \to & \mathrm{GL}((\mathbb{R}^k)^*) & \to & \mathrm{GL}((\mathbb{R}^k)^*) \\ \\ A & \mapsto & A^* & & \\ & & B & \mapsto & B^{-1} \end{array}$$

which is C^∞. We'll see next time that the smoothness of the purported transition maps allows us to make E^* into a vector bundle.

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