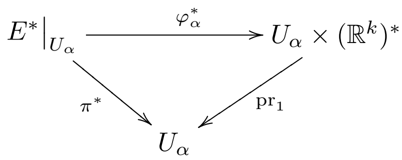
*Last Time.*

(1) Given a vector bundle $E \xrightarrow{\pi} M$ we started constructing the dual bundle $E^* \xrightarrow{\pi^*} M$ as a set $E^* = \coprod_{q \in M} (E_q)^*$.
(2) Out of trivializations $\varphi_\alpha : E\big|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ we constructed purported trivializations $\varphi_\alpha^* : E^*\big|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^k)^*$ bijections, linear on each fiber.
(3) We checked $(\varphi_\alpha^* \circ (\varphi_\beta^*)^{-1})(q, l) = (q, \varphi_{\alpha\beta}^*(q) l)$ where $\varphi_{\alpha\beta}^* : U_\alpha \cap U_\beta \to \mathrm{GL}((\mathbb{R}^k)^*)$ are $C^\infty$.

To prove that $E^*$ is a manifold, that $\varphi_\alpha^*$ are diffeomorphisms, and that $\pi^* : E^* \to M$ is smooth we need a proposition.

**Proposition 27.1.** *Suppose that we have a set $X$, a cover $\{U_\alpha\}_{\alpha \in A}$ of $X$, a collection of bijections $\{\psi_\alpha \mid V_\alpha \to W_\alpha\}_{\alpha \in A}$ where $W_\alpha$ are manifolds such that for all $\alpha, \beta \in A$*
*(i) $\psi_\alpha(V_\alpha \cap V_\beta)$ is open in $W_\alpha$ and*
*(ii) $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta)$ is $C^\infty$,*
*then $X$ is a manifold so that all $\psi_\alpha$ are diffeomorphisms.*

Note that the proposition implies that the total space $E^*$ of the bundle dual to $E \to M$ is a manifold. Moreover, for all $U_\alpha$ the following diagram commutes




Hence $\pi^*\big|_{E^*_{U_\alpha}} = \mathrm{pr}_1 \circ \varphi_\alpha^*$ is $C^\infty$. Therefore $\pi^* : E^* \to M$ is $C^\infty$. Not hard to check that $\varphi_\alpha^* : \big|_{E^*_{U_\alpha}} \to U_\alpha \times (\mathbb{R}^k)^*$ are diffeomorphisms. Consequently $E^* \xrightarrow{\pi^*}$ is indeed a vector bundle.

*Sketch of proof.*

(1) The sets $\{\varphi_\alpha^{-1}(\mathcal{O}) \mid \alpha \in A \text{ and } \mathcal{O} \in W_\alpha \text{ is open}\}$ form a basis for a topology on $X$ which make $\psi_\alpha$ into homeomorphisms.
(2) Each point $x \in X$ lies in some $V_\alpha$. $\psi_\alpha(x)$ lies in a coordinate chart $\varphi : U \to \mathbb{R}^m$ on $W_\alpha$. Declare $\varphi \circ \psi_\alpha : \psi_\alpha^{-1}(U) \to \mathbb{R}^m$ to be a coordinate chart. *(ii)* implies that the charts define an atlas.

□

Can we perform other operations? What do we need?

**Example 27.2.** Suppose given a vector bundle $E \xrightarrow{\pi} M$ of rank $k$ we want to construct the $n^{th}$ exterior power $\Lambda^n E \xrightarrow{\tau} M$ of a vector bundle $E \to M$. We set

$$\Lambda^n E = \coprod_{q \in M} \Lambda^n(E_q) \quad \text{(as a set)}.$$

Out of a collection $\{\varphi_\alpha : E\big|_\alpha \to U_\alpha \times V\}_{\alpha \in A}$ of local trivializations with $\bigcup U_\alpha = M$ ($V$ is a fixed finite dimensional vector space) we get for all $\alpha$ and all $q \in U_\alpha$ linear isomorphisms

$$\varphi_\alpha\big|_{E_q} : E_q \xrightarrow{\sim} \{q\} \times V.$$

Applying exterior power $\Lambda^n$ to everything above we get

$$\Lambda^n(\varphi_\alpha\big|_{E_q}) : \Lambda^n E_q \to \{q\} \times \Lambda^n(V),$$

whence

$$\Lambda^n(\varphi_\alpha) : \Lambda^n E_q\big|_{U_\alpha} \to \{U_\alpha\} \times \Lambda^n(V)$$

Hence for all indices $\alpha$ and $\beta$ with $U_\alpha \cap U_\beta \neq \emptyset$ we have

$$\left(\Lambda^n \varphi_\alpha \circ (\Lambda^n \varphi_\beta)^{-1}\right)(q, \eta) = (q, \Lambda^n(\varphi_{\alpha\beta}(q))\eta).$$

For any finite dimensional vector space $V$ over $\mathbb{R}$ we have a map

$$\Lambda^n : \mathrm{GL}(V) \to \mathrm{GL}(\Lambda^n V), \quad A \mapsto \Lambda^n A,$$

which is a group homomorphism and is *polynomial* in $A$. That is to say, $\Lambda^n((a_{ij}))$ has entries which are polynomials in $a_{ij}$'s. Hence $\Lambda^n$ is $C^\infty$. Therefore the purported transition maps $\Lambda^n(\varphi_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{GL}(\Lambda^n V)$ are $C^\infty$. Now Proposition 27.1 implies that $\Lambda^n E$ is a manifold and the local trivializations $\{\Lambda^n \varphi_\alpha : \Lambda^n E\big|_{U_\alpha} \to \{U_\alpha\} \times \Lambda^n(V)$ are smooth. Proceeding as in the case of the dual bundle we get that $\Lambda^n E \xrightarrow{\tau} M$ is a vector bundle of rank $\binom{k}{n}$.

Note that at this point we have constructed, for any manifold $M$, the bundles $\Lambda^n(T^*M) \to M$ and hence differential forms.

**Example 27.3.** Suppose that $E \xrightarrow{\pi_E} M$ and $F \xrightarrow{\pi_F} M$ are two vector bundles. Let's try and construct the *Whitney sum* $E \oplus F \to M$. We choose a cover $U_\alpha$ of $M$ such that $E\big|_{U_\alpha}$ and $F\big|_{U_\alpha}$ are both trivial for all $\alpha$. We have trivializations

$$\varphi_\alpha^E : E\big|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$$
$$\varphi_\alpha^F : F\big|_{U_\alpha} \to U_\alpha \times \mathbb{R}^l$$

We set $E \oplus F = \coprod_{q \in M} E_q \oplus F_q$ (as a set). The purported trivializations are

$$\varphi_\alpha^E \oplus \varphi_\alpha^F : E\big|_{U_\alpha} \oplus F\big|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^k \oplus \mathbb{R}^l)$$

The corresponding transition maps are

$$\varphi_{\alpha\beta}^{E\oplus F}(q) = \varphi_{\alpha\beta}^E(q) \oplus \varphi_{\alpha\beta}^F(q)$$

and the map $\mathrm{GL}(\mathbb{R}^k) \times \mathrm{GL}(\mathbb{R}^l) \to \mathrm{GL}(\mathbb{R}^k \oplus \mathbb{R}^l)$ with $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is $C^\infty$. Proceeding as in the case of exterior powers we get that $E \oplus F \to M$ is a vector bundle.

*Question.* What's the general principle?

*Answer.* $C^\infty$ functors.

To define *functors* we must first define *categories*.

**Definition 27.4.** A *category* $\mathcal{C}$ consists of

- A collection of objects $\mathcal{C}_0$.
- For each pair of objects $X, Y \in \mathcal{C}_0$ a set $\mathrm{Hom}_\mathcal{C}(X, Y)$ of arrows/morphisms.
- For each triple of objects $X, Y, Z \in \mathcal{C}_0$ a composition

$$\begin{array}{rcl} \circ : \mathrm{Hom}_\mathcal{C}(Y, Z) \times \mathrm{Hom}_\mathcal{C}(X, Y) & \to & \mathrm{Hom}_\mathcal{C}(X, Z) \\ \left(Z \xleftarrow{g} Y, Y \xleftarrow{f} X\right) & \mapsto & Z \xleftarrow{g\circ f} X \end{array}$$

- For each object $X \in \mathcal{C}_0$ a morphism $1_X \in \mathrm{Hom}_\mathcal{C}(X, Y)$ such that
  (i) For all $f \in \mathrm{Hom}_\mathcal{C}(X, Y)$ we have $1_Y \circ f = f = f \circ 1_X$; and
  (ii) $\circ$ is associative: for all $W \xleftarrow{h} Z \xleftarrow{g} Y \xleftarrow{f} X$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

We set $\mathcal{C}_1 = \coprod_{X, Y \in \mathcal{C}_0} \mathrm{Hom}_\mathcal{C}(X, Y)$. This is a collection of all morphisms.

**Example 27.5.** $\mathcal{C} = \mathrm{Set}$, the collection of all sets and maps of sets is a category. $\mathcal{C}_0$ is the collection of all sets and for all $X, Y \in \mathcal{C}_0$ we have $\mathrm{Hom}_{\mathrm{Set}}(X, Y) = \{f : X \to Y \mid f \text{ is a function}\}$

**Example 27.6.** $\mathcal{C} = \mathrm{Top}$, the category of topological spaces and continuous maps.

**Example 27.7.** $\mathcal{C} = \mathrm{Man}$, the category of manifolds. $\mathcal{C}_0$ is usually taken as the collection of all finite dimensional, Hausdorff, paracompact manifolds and the morphisms are $C^\infty$ maps.

**Example 27.8.** $\mathcal{C} = \mathrm{Lie}$, the category of Lie groups.

**Example 27.9.** $\mathcal{C} = \mathrm{Vec}$, the category of finite dimensional vector spaces over $\mathbb{R}$ where the morphisms are linear maps.

**Example 27.10.** $\mathcal{C} = \mathrm{Vec}^{\mathrm{iso}}$, the category of finite dimensional vector spaces over $\mathbb{R}$ where the morphisms are linear isomorphisms.

*Next Time.* Functors, smooth functors, and differential forms.

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