Last Time.

(1) Given a vector bundle $E \to M$ we've constructed the dual bundle $E^* \to M$ and exterior powers bundles, $\Lambda^n E \to M$, $n = 0, \dots,$rank E. Note that for any manifold M we have $TM \to M$ and hence we also have
   - the dual bundle $(TM)^* = T^*M$, which is the cotangent bundle.
   - $\Lambda^n(T^*M)$ for $0 \le n \le \dim M$.
   - $\Gamma(\Lambda^n(T^*M)) = \Omega^n(M)$, differential forms.

(2) We've constructed the *Whitney sum* $E \oplus F \to M$ of vector bundles$E \to M$ and $F \to M$.

(3) We've also defined categories and considered a few examples of categories: the category Set of sets, the category Top of topological spaces and continuous maps, the category Man of manifolds and smooth maps, the category Vec of real finite dimensional vector spaces and linear maps, the category $\mathsf{Vec}^{\mathsf{iso}}$ of finite dimensional real vector spaces and linear *isomorphisms.* Note that for an object V of $\mathsf{Vec}^{\mathsf{iso}}$, i.e., for a vector space V, we have $\mathrm{Hom}_{\mathsf{Vec}^{\mathsf{iso}}}(V, V) = \mathrm{GL}(V)$.

Definition 28.1. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of objects $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and (compatible) maps $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following holds

(1) For all objects $A \in \mathcal{C}_0$ we have $F_1(1_A) = 1_{F_0(A)}$.

(2) For all (compatible) morphisms $f, g \in \mathcal{C}_1$ we have $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

Example 28.2. We have the *underlying functor* $U : \mathsf{Man} \to \mathsf{Set}$ which assigns to a manifold the underlying set.

Remark 28.3. A *contravariant functor* reverses the composition order. So instead of condition (2) in Definition 28.1 we have for all (compatible) morphisms $f, g \in \mathcal{C}_1$ that $F_1(g \circ f) = F_1(f) \circ F_1(g)$.

Notation. Given a functor $F : \mathcal{C} \to \mathcal{D}$ one usually writes F for both F_1 and F_0.

Example 28.4. The functor $(-)^* : \mathsf{Vec} \to \mathsf{Vec}$ that takes the duals, that is, $(V \xrightarrow{A} W) \mapsto (V^* \xleftarrow{A^*} W^*)$ is a contravariant functor.

Example 28.5. $[(-)^*]^{-1} : \mathsf{Vec}^{\mathsf{iso}} \to \mathsf{Vec}^{\mathsf{iso}}$ with $(V \xrightarrow{A} W) \mapsto (V^* \xrightarrow{(A^*)^{-1}} W^*)$ is a covariant functor.

Example 28.6. The mapping $\Lambda^n : \mathsf{Vec} \to \mathsf{Vec}$ with $(A \xrightarrow{A}) \mapsto (\Lambda^n V \xrightarrow{\Lambda^n A} \Lambda^n W)$ is a functor.

Question. What about $\oplus$ and $\otimes$? They don't look like functors since they require *pairs* of vector spaces as inputs and *pairs* of maps.

Definition 28.7. Let $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(n)}$ be categories. The objects of the product category $\mathcal{C}^{(1)} \times \cdots \times \mathcal{C}^{(n)}$ are n-tuples of objects $(X_1, \dots, X_n)$ such that each $X_i \in \mathcal{C}^{(i)}$. Morphisms are n-tuples of morphisms of $\mathcal{C}^{(i)}$'s. I.e $(X_1 \xrightarrow{f_1} Y_1, \dots, X_n \xrightarrow{f_n} Y_n)$ where each $X_i \xrightarrow{f_i} Y_i$ is an arrow in $\mathcal{C}^{(i)}$.

Example 28.8. $\oplus : \mathsf{Vec} \times \mathsf{Vec} \to \mathsf{V}$ with $(V_1 \xrightarrow{A_1} W_1, V_2 \xrightarrow{A_2} W_2) \mapsto (V_1 \oplus V_2 \xrightarrow{A_1 \oplus A_2} W_1 \oplus W_2)$ is a functor.

Example 28.9 (Smooth functor)**.** $\otimes : \mathsf{Vec} \times \mathsf{Vec} \to \mathsf{V}$ with $(V_1 \xrightarrow{A_1} W_1, V_2 \xrightarrow{A_2} W_2) \mapsto (V_1 \otimes V_2 \xrightarrow{A_1 \otimes A_2} W_1 \otimes W_2)$ is a functor.

Definition 28.10. A (covariant) functor $F : (\mathsf{Vec}^{\mathsf{iso}})^n \to \mathsf{Vec}^{\mathsf{iso}}$ is C^∞ if for any n-tuple of vector spaces $(V_1, \dots, V_n)$ the map $F : \underbrace{\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)}_{\mathrm{Hom}_{(\mathrm{Vec}^{\mathrm{iso}})^n}((V_1,\dots,V_n),(V_1,\dots,V_n))} \to \underbrace{\mathrm{GL}(F(V_1, \dots, V_n))}_{\mathrm{Hom}_{\mathrm{Vec}^{\mathrm{iso}}}(F(V_1,\dots,V_n),F(V_1,\dots,V_n))}$ is C^∞.

Example 28.11. Examples 28.6, 28.8, and 28.9 above provide C^∞ functors if we restrict Vec to $\mathsf{Vec}^{\mathsf{iso}}$.

Theorem 28.12. *If $F : (\mathsf{Vec}^{\mathsf{iso}})^n \to \mathsf{Vec}^{\mathsf{iso}}$ is a C^∞ functor, then for any n-tuple of vector bundles $\{E_i \to M\}_{i=1}^n$ there exists a vector bundle $F(E_1, \dots, E_n) \to M$ such that for all $q \in M$ we have $F(E_1, \dots, E_n)_q = F((E_1)_q, \dots, (E_n)_q)$.*

Sketch of Proof. As a set define $F(E_1, \dots, E_n) \stackrel{\text{def}}{=} \coprod_{q \in M} F((E_1)_q, \dots, (E_n)_q)$. Let $\{\varphi_\alpha^{(i)} : E^{(i)}\big|_{U_\alpha} \to U_\alpha \times \mathbb{R}^{k_i}\}_{\alpha \in A}{}_{i=1}^{n}$ be a collection of trivializations with $\{U_\alpha\}_{\alpha \in A}$ a cover of M. Then

$$F\left(\varphi_\alpha^{(1)}, \dots, \varphi_\alpha^{(n)}\right) : F(E_1, \dots, E_n)\big|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n})$$

are purported trivializations of $F(E_1, \dots, E_n)$. Then for all $q \in U_\alpha$ we have

$$F\left(\varphi_\alpha^{(1)}, \dots, \varphi_\alpha^{(n)}\right)|_{F((E_1)_q, \dots, (E_n)_q)} = F\left(\varphi_\alpha^{(1)}(q), \dots, \varphi_\alpha^{(n)(q)}\right)$$

The corresponding transition maps of $F(E_1, \dots, E_n)$ are

$$\varphi_{\alpha\beta}^{F(E_1, \dots, E_n)}(q) = F\left(\varphi_{\alpha\beta}^{(1)}(q), \dots, \varphi_{\alpha\beta}^{(n)}(q)\right)$$

which are C^∞ since F is a C^∞ functor. □

Our next goal, for a week or two are Stokes' and divergence theorems. Recall the fundamental theorem of calculus. It states that

$$\int_a^b f'(x)\, \mathrm{d}x = f(b) - f(a)$$

which we may rewrite as

$$\int_{[a,b]} \mathrm{d}f = \int_{\partial([a,b])} f,$$

Recall also

Theorem 28.13 (Green's Theorem)**.** *If $D \subseteq \mathbb{R}^2$ is a domain with smooth boundary ∂D, then*

$$\int_{\partial D} P(x,y)\, \mathrm{d}x - Q(x,y)\, \mathrm{d}y = \int_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y$$

Note that

$$\begin{aligned} \mathrm{d}P &= \frac{\partial P}{\partial x}\, \mathrm{d}x + \frac{\partial P}{\partial y}\, \mathrm{d}y \\ \mathrm{d}Q &= \frac{\partial Q}{\partial x}\, \mathrm{d}x + \frac{\partial Q}{\partial y}\, \mathrm{d}y \end{aligned}$$

Hence, if we set $\alpha = P(x,y)\, \mathrm{d}x - Q(x,y)\, \mathrm{d}y$, we can rewrite the statement of Green's theorem as

$$\int_{\partial D} \alpha = \int_D \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y$$

If we set $\mathrm{d}\alpha := \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y$, then the statement of Green's theorem shortens to

$$\int_{\partial D} \alpha = \int_D \mathrm{d}\alpha,$$

which now looks just like the fundamental theorem of calculus except now instead of a function f we have a one-form α. The general theorem that subsumes the two theorems above as special cases is:

Theorem 28.14 (Stokes' Theorem)**.** *Let D be a domain in an oriented manifold M with boundary ∂D (oriented appropriately). Then for all $\omega \in \Omega_c^{\dim M - 1}(M)$*

$$\int_{\partial D} \omega = \int_D \mathrm{d}\omega.$$

To makes sense of the statement we need to sort out a number of things: "domain," why the boundary of a domain is a manifold, "induced orientation of the boundary," the exterior derivative d applied to all differential forms and not just to functions ...

Typeset by R. S. Kueffner II