Last Time. We started discussing Stokes' theorem:

$$\int_{\partial D} \alpha = \int_D \mathrm{d}\alpha$$

for a regular domain D with boundary ∂D in an oriented manifold M and a compactly supported form α of degree $\dim M - 1$.

Today we will make sense of the "d" in "$\mathrm{d}\alpha$"in the statement of Stokes' theorem above. Later we will define regular domains, their boundaries and orientation of boundaries induced by the orientation of the ambient manifold.

Theorem 29.1 (Exterior Derivative)**.** *For every manifold M there is a unique $\mathbb{R}$-linear map*

$$\mathrm{d}_M : \Omega^*(M) \to \Omega^{*+1}(M)$$

(i.e. $\forall \omega \in \Omega^k(M)$ we have $\mathrm{d}_M \omega \in \Omega^{k+1}(M)$) called exterior derivative *such that*

1. *For all $f \in C^\infty = \Omega^0(M)$ we have $\mathrm{d}_M f = \mathrm{d}f$.*
2. *For all open $W \subseteq M$ and all $\omega \in \Omega^k(M)$ we have $(\mathrm{d}_M\omega)\big|_W = \mathrm{d}_W(\omega\big|_W)$*
3. *For all $\omega \in \Omega^k(M)$ and all $\eta \in \Omega^l(M)$ we have $\mathrm{d}_M(\omega \wedge \eta) = \mathrm{d}_M(\omega) \wedge \eta + (-1)^k \omega \wedge \mathrm{d}_M \eta$*
4. *For all $\omega \in \Omega^k(M)$ we have $\mathrm{d}_M(\mathrm{d}_M\omega) = 0$ (i.e. $\mathrm{d}_M^2 = 0$).*

Proof of Uniqueness. Suppose that there exists $\mathrm{d}_M : \Omega^*(M) \to \Omega^{*+1}(M)$ with properties (1)-(4) above. Let $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart. Then for all $\alpha \in \Omega^k(M)$

$$\alpha\big|_U = \sum_{|I|=k} a_I \, \mathrm{d}x_I = \sum_{I=i_1<\cdots<i_k} a_I \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

Claim 1. *If d exists then we must have $\mathrm{d}_U(\mathrm{d}x_I) = 0$*

Proof. We will proceed by induction on k. If $k = 1$ then

$$\mathrm{d}_U(\mathrm{d}x_{i_1}) \overset{by(1)}{=} \mathrm{d}_U \, \mathrm{d}_U x_{i_1} \overset{by(4)}{=} 0.$$

Suppose that $\mathrm{d}_U(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}) = 0$, then

$$\mathrm{d}_U(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n} \wedge \mathrm{d}x_{i_{n+1}}) \overset{by(3)}{=} \mathrm{d}_U(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}) \wedge \mathrm{d}x_{i_{n+1}} + (-1)^n \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n} \wedge \mathrm{d}_U(\mathrm{d}x_{i_{n+1}}).$$

Hence

$$\begin{aligned}
(\mathrm{d}_M\alpha)\big|_U &= \mathrm{d}_U(\alpha\big|_U) \\
&= \mathrm{d}_U\left(\sum_{|I|=k} a_I \, \mathrm{d}x_I\right) \\
&= \sum_{|I|=k} (\mathrm{d}_U a_I \wedge \mathrm{d}x_I + a_I \, \mathrm{d}_U(\mathrm{d}x_I)) \\
&= \sum_{|I|=k} \mathrm{d}a_I \wedge \mathrm{d}x_I.
\end{aligned}$$

Therefore if $\mathrm{d}'_M \Omega^*(M) \to \Omega^{*+1}(M)$ is another $\mathbb{R}$-linear map with properties (1)-(4), then for all k, all $\alpha \in \Omega^k(M)$, and all coordinate charts U we have

$$(\mathrm{d}'_M\alpha)\big|_U = \mathrm{d}'_U(\alpha\big|_U) = \mathrm{d}'_U\left(\sum a_I \, \mathrm{d}x_I\right) = \sum \mathrm{d}a_I \wedge \mathrm{d}x_I = (\mathrm{d}_M\alpha)\big|_U.$$

This proves uniqueness of the exterior derivative d_M. □

Proof of Existence. For each coordinate chart $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M define for all k the map $\mathrm{d}_U : \Omega^k(M) \to \Omega^{k+1}(M)$ by

$$\mathrm{d}_U\left(\sum_{|I|=k} a_I \, \mathrm{d}x_I\right) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{d}a_I & \text{if } |I| = 0 \\ \sum_{|I|=k} \mathrm{d}a_I \wedge \mathrm{d}x_I & \text{if } |I| > 0 \end{cases} \tag{29.1}$$

Assume for the moment:

Claim 2. d_U *defined by* (29.1) *has properties (1)-(4).*

Now define $\mathrm{d}_M : \Omega^*(M) \to \Omega^{*+1}(M)$ as follows: for any coordinate chart U set

$$(\mathrm{d}_M\alpha)\big|_U = \mathrm{d}_U(\alpha\big|_U) \tag{29.2}$$

Does equation 29.2 make sense? Suppose that U and V are two compatible coordinate charts. Then

$$\begin{aligned}(\mathrm{d}_U(\alpha\big|_U))\big|_{U\cap V} &\overset{Claim\ (2)}{=} \mathrm{d}_{U\cap V}(\alpha\big|_U\big|_{U\cap V})\\ &= \mathrm{d}_{U\cap V}(\alpha\big|_V\big|_{U\cap V})\\ &\overset{Claim\ (2)}{=} (\mathrm{d}_U(\alpha\big|_V))\big|_{U\cap V}\end{aligned}$$

Next we prove claim 2. We check conditions (1)-(4).

(1). For all $f \in C^\infty(U)$ we have $\mathrm{d}_U f = \mathrm{d}f$ by definition of d_U.

(2). Recall that we proved that d commutes with pullbacks and restrition $\big|_W$ to W is the pullback by the inclusion $W \hookrightarrow U$. Hence for any $f \in C^\infty(U)$ and any open $W \subseteq U$ we have

$$(\mathrm{d}_U f)\big|_W = \mathrm{d}f\big|_W = \mathrm{d}(f\big|_W) = \mathrm{d}_W(f\big|_W)$$

Now

$$\begin{aligned}\mathrm{d}_U(a_I\,\mathrm{d}x_I)\big|_W &= (\mathrm{d}a_I \wedge \mathrm{d}x_I)\big|_W\\ &= \mathrm{d}a_I\big|_W \wedge (\mathrm{d}x_I)\big|_W\\ &= \mathrm{d}(a_I\big|_W) \wedge \mathrm{d}(x_I\big|_W)\\ &= \mathrm{d}_W\left((a_I\,\mathrm{d}x_I)\big|_W\right)\end{aligned}$$

Hence d_U has property (2).

(3). It is no loss of generality to assume that $\omega = a_I\,\mathrm{d}x_I$ and $\eta = b_J\,\mathrm{d}x_J$ for some indicies I and J and some functions a_I, b_J. Then

$$\begin{aligned}\mathrm{d}_U(\omega\wedge\eta) &= \mathrm{d}_U(a_I\,\mathrm{d}x_I \wedge b_J\,\mathrm{d}x_J)\\ &= \mathrm{d}_u(a_I b_J\,\mathrm{d}x_I \wedge b_J\,\mathrm{d}x_J)\\ &= \mathrm{d}a_I b_J \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J\\ &= \underbrace{\mathrm{d}a_I \wedge \mathrm{d}x_I}_{\mathrm{d}\omega} \wedge \underbrace{(b_J\,\mathrm{d}x_J)}_{\eta} + (-1)^k \underbrace{(a_I\,\mathrm{d}x_I)}_{\omega} \wedge \underbrace{\mathrm{d}b_J \wedge \mathrm{d}x_J}_{\mathrm{d}\eta}\end{aligned}$$

This proves that (3) holds for d_U.

(4).

$$\begin{aligned}\mathrm{d}_U\left(\mathrm{d}_U(a_I\,\mathrm{d}x_I)\right) &= \mathrm{d}_U(\mathrm{d}a_I \wedge \mathrm{d}x_I)\\ &= \mathrm{d}_U(\mathrm{d}a_I) \wedge \mathrm{d}x_I + (-1)^k\,\mathrm{d}a_I \wedge \underbrace{\mathrm{d}_U(\mathrm{d}x_I)}_{=0}\\ &= \mathrm{d}_U\left(\sum \frac{\partial a_I}{\partial x_i}\,\mathrm{d}x_i\right) \wedge \mathrm{d}x_I\end{aligned}$$

Now

$$\mathrm{d}_U\left(\sum \frac{\partial a_I}{\partial x_i}\,\mathrm{d}x_i\right) = \sum_{i,j} \underbrace{\frac{\partial^2 a_I}{\partial x_i \partial x_j}}_{Symmetric} \underbrace{\mathrm{d}x_j \wedge \mathrm{d}x_i}_{Skew-symmetric},$$

hence is 0. Thefore $\mathrm{d}_U\left(\mathrm{d}_U(a_I\,\mathrm{d}x_I)\right) = 0$ and consequently $\mathrm{d}_U \circ \mathrm{d}_U = 0$. This proves claim 2. □

It remains to show that d_M defined by equation 29.2 has propeties (1)-(4).
(1). For all $f \in C^\infty(M)$ and any cooridante chart U

$$(\mathrm{d}_M f)\big|_U \overset{\text{by our definition of } \mathrm{d}_M}{=} \mathrm{d}_U(f\big|_U) = \mathrm{d}(f\big|_U) = \mathrm{d}f\big|_U.$$

Since U is arbitrary, $\mathrm{d}_M f = \mathrm{d}f$.

(2). For all open $W \subseteq M$ and all coordinate charts $U \subseteq M$ we know that $U \cap W$ is a coordinate chart on W. Since d_U has property (3), we know that for all $\mu \in \Omega^*(U)$

$$(\mathrm{d}_U \mu)\big|_{U\cap W} = \mathrm{d}_{U\cap W}(\mu\big|_{U\cap W}) \tag{29.3}$$

Therefore for any $\omega \in \Omega^*(M)$

$$\begin{aligned} \left((\mathrm{d}_M\omega)\big|_W\right)\big|_{U\cap W} &= \mathrm{d}_M\omega\big|_{U\cap W} \\ &= \left((\mathrm{d}_M\omega)\big|_U\right)\big|_{W\cap U} \\ &\overset{29.2}{=} \left(\mathrm{d}_U(\omega\big|_U)\right)\big|_{W\cap U} \\ &\overset{29.3}{=} \mathrm{d}_{W\cap U}(\omega\big|_U\big|_{W\cap U}) \\ &\overset{29.2}{=} \mathrm{d}_W(\omega\big|_W)\big|_{W\cap U} \end{aligned}$$

Hence $\mathrm{d}_M\omega\big|_W = \mathrm{d}_W(\omega\big|_W)$ if d_M and d_W are defined by equation 29.2.

(3). Say $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ and U a coordinate chart, then

$$\begin{aligned} \mathrm{d}_M(\omega\wedge\eta)\big|_U &\overset{29.2}{=} \mathrm{d}_U\left((\omega\wedge\eta)\big|_U\right) \\ &= \mathrm{d}_U\left(\omega\big|_U \wedge \eta\big|_U\right) \\ &= \mathrm{d}_U\left(\omega_U\right) \wedge \left(\eta\big|_U\right) + (-1)^k\left(\omega\big|_U\right) \wedge \mathrm{d}_U\left(\eta\big|_U\right) \\ &= \left((\mathrm{d}_M\omega)\wedge\eta\right)\big|_U + (-1)^k\left(\omega\wedge(\mathrm{d}_M\eta)\right)\big|_U \end{aligned}$$

(4).

$$\mathrm{d}_M(\mathrm{d}_M\omega)\big|_U = \mathrm{d}_U\left((\mathrm{d}_M\omega)\big|_U\right) = \mathrm{d}_U\left(\mathrm{d}_U(\omega\big|_U)\right) = 0$$

since $\mathrm{d}_U \circ \mathrm{d}_U = 0$.

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