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U of Wisc.

Non compact symplectic toric manifolds.

This is joint work with Yael Karshon. ①

Consider a manifold M with a symplectic form ω. Then any function (Hamiltonian) $\mu: M \to \mathbb{R}$ defines a vector field, hence a flow $\{\varphi_t\}$.

I'll call $\mu \in C^\infty(M)$ <u>periodic</u> if $\varphi_{t+1} = \varphi_t$ $\forall t$, ie $\{\varphi_t\}$ is periodic of period 1.

Then $\{\varphi_t\}$ defines an action of $S^1 = \mathbb{R}/\mathbb{Z}$ on M:

$$[t]\cdot m := \varphi_t(m) \qquad \forall [t] \in \mathbb{R}/\mathbb{Z}.$$

Next suppose we have $n = \frac{1}{2}\dim M$ periodic Hamiltonians $\mu_1, \dots, \mu_n$ with $\{\mu_i, \mu_j\} = 0$ (Poisson bracket)

Then the flows $\varphi^1_t, \dots, \varphi^n_t$ commute and

$$\mu = (\mu_1, \dots, \mu_n): M \to \mathbb{R}^n \qquad \text{(moment map)}$$

defines an action of $(\mathbb{R}/\mathbb{Z})^n \simeq \mathbb{R}^n/\mathbb{Z}^n \equiv \mathbb{T}^n$:

$$[t_1, \dots, t_n]\cdot m = (\varphi^1_{t_1} \circ \dots \circ \varphi^n_{t_n})(m),$$

We call this action of $\mathbb{T}^n$ on (M, ω) Hamiltonian.

The triple $(M, \omega, \mu: M \to \mathbb{R}^n)$ is called a <u>symplectic $\mathbb{T}^n$-toric manifold</u>. if $n = \frac{1}{2}\dim M$ and $d\mu_1 \wedge \dots \wedge d\mu_n \neq 0$ on an open dense set (ie., action of $\mathbb{T}^n$ is effective).

<u>Facts</u>
- Connected components of fibers of μ are $\mathbb{T}^n$ orbits
- The orbits of $\mathbb{T}^n$ are isotropic

    dim count $\Rightarrow$ generic fibers are Lagrangian.

Moment maps for Hamiltonian torus actions on compact connected manifolds have remarkable properties.

<u>Thm</u> (Atiyah, Guillemin-Sternberg)
($M, \omega, \mu: M \to \mathbb{R}^k$) Hamiltonian $\mathbb{T}^k$ action on a compact conn. symplectic manifold. Then

1) $\mu(M)$ is a polytope [convexity]
2) fibers of μ are connected. [connectedness]

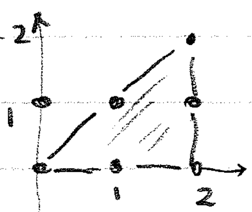
The case of symplectic toric manifolds is even more rigid.

<u>Thm</u> (Delzant) Let ($M, \omega, \mu: M \to \mathbb{R}^n$) be a compact connected symplectic $\mathbb{T}^n$-toric manifold. Then

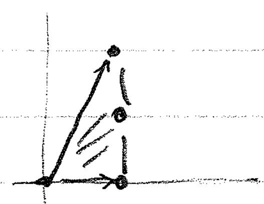
1) $\mu(M)$ is a unimodular ("Delzant") polytope
2) Unimodular polytopes in $\mathbb{R}^n$ classify equivalence classes of (compact connected) symplectic $\mathbb{T}^n$-toric manifolds.

$\Delta \subseteq \mathbb{R}^n$ is <u>unimodular</u> if $\forall$ vertex p of Δ $\exists$ a basis $\{v_1 \ldots v_n\}$ of $\mathbb{Z}^n$ so that edges of Δ coming out of p lie on rays $p + tv_i$, $1 \le i \le n$, $t \ge 0$.

<u>Ex</u>



unimodular



not unimodular: $\{(1,0), (1,2)\}$ is not a basis of $\mathbb{Z}^n$.

Note that Delzant's theorem has two parts:

<u>Existence</u>: given a unimodular polytope $\Delta \subset \mathbb{R}^n$ $\exists$ [compact connected] symplectic $\mathbb{T}^n$-toric manifold ($M, \omega, \mu: M \to \mathbb{R}^n$) with $\mu(M) = \Delta$

<u>Uniqueness</u> if (M_i, ω_i, μ_i), $i = 1, 2$ are two symplectic $\mathbb{T}^n$-toric compact connected manifolds with $\mu_1(M_1) = \mu_2(M_2)$ Then $\exists$ T^n-equivariant symplecto $\varphi: M_1 \to M_2$ with

$$\mu_2 \circ \varphi = \mu_1$$

<u>Note also:</u> At the heart of the proof of Aiyah, Guillemin-Sternberg Thm is Bott-Morse theory: if $h: M \to \mathbb{R}$ is a periodic Hamiltonian, then h is Bott-Morse with all indices even.

Now suppose $(M, \omega, \mu: M \to \mathbb{R}^n)$ is symplectic $\mathbb{T}^n$-toric
M connected but <u>not compact.</u>
Things look hopeless —
- no Morse theory
- fibers of μ need not be connected
- $\mu(M)$ tells you little.

However, if we change point of view things look better.

<u>Facts</u>
- $M/\mathbb{T}^n$ is a manifold with corners
- $\mu: M \to \mathbb{R}^n$ descends to $\bar{\mu}: M/\mathbb{T}^n \to \mathbb{R}^n$, which is locally a <u>unimodular embedding</u> — $\bar{\mu}$ maps corners of $M/\mathbb{T}^n$ to unimodular cones.

orbital moment map

<u>Thm</u> (Karshon-L) Let $(M, \omega, \mu: M \to \mathbb{R}^n)$ be a (connected) symplectic $\mathbb{T}^n$-toric manifold. Then

"Existence" Given a unimodular local embedding of a manifold with corners $\psi: W \to \mathbb{R}^n$, there exists a symplectic $\mathbb{T}^n$-toric manifold (M, ω, μ) with $M/\mathbb{T}^n = W$ and $\bar{\mu} = \psi$.

"Uniqueness" The equivalence classes of symplectic toric manifolds (M, ω, μ) with $M/\mathbb{T}^n = W$ and $\bar{\mu} = \psi$ are in bijective correspondence with the elements of $H^2(W, \mathbb{Z}^n \times \mathbb{R})$.

Comments

"Uniqueness" is not hard with existing technology.

"Existence" is harder, in contrast to the compact case (Delzant's thm).

Delzant constructs toric manifolds from a polytope Δ as symplectic quotients. In the non-compact case there is no polytope.

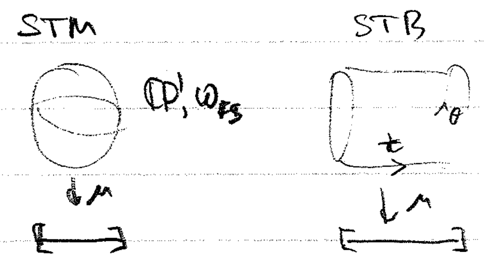
So we needed a new idea.

Fix a unimodular local embedding $\psi : W \to \mathbb{R}^n$ of an n-dim manifold with corners.

We have a category $STM(W \xrightarrow{\psi} \mathbb{R}^n)$ of symplectic $\mathbb{T}^n$-toric manifolds with orbit space W and orbital moment map ψ.

We consider a new category $STB(W \xrightarrow{\psi} \mathbb{R}^n)$ of principal $\mathbb{T}^n$-bundles (with corners) over W with symplectic forms and orbital moment map ψ.

Ex $\quad W = [\!-\!-\!] \xhookrightarrow{\psi} \mathbb{R}$



$\omega = d(t\, d\theta)$

Theorem (Karshon - L). There is a functor $STB \xrightarrow{c} STM$, which is an equivalence of categories. Since $STB \neq \emptyset$, $STM \neq \emptyset$ as well.

In fact more is true: $\boxed{\text{STB is a monoidal category and STM is an STB-torsor.}}$

Note: This explains the bijection

iso classes of objects in $STM(W) \longleftrightarrow H^2(W, \mathbb{Z}^n \times \mathbb{R})$; 2-forms)

We have a bijection: iso classes of objects in $STB \leftrightarrow H^2(W, \mathbb{Z}^n \times \mathbb{R})$

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